Iterative Methods over the Centuries Part II: Linear Systems of Equations

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Iterative Methods
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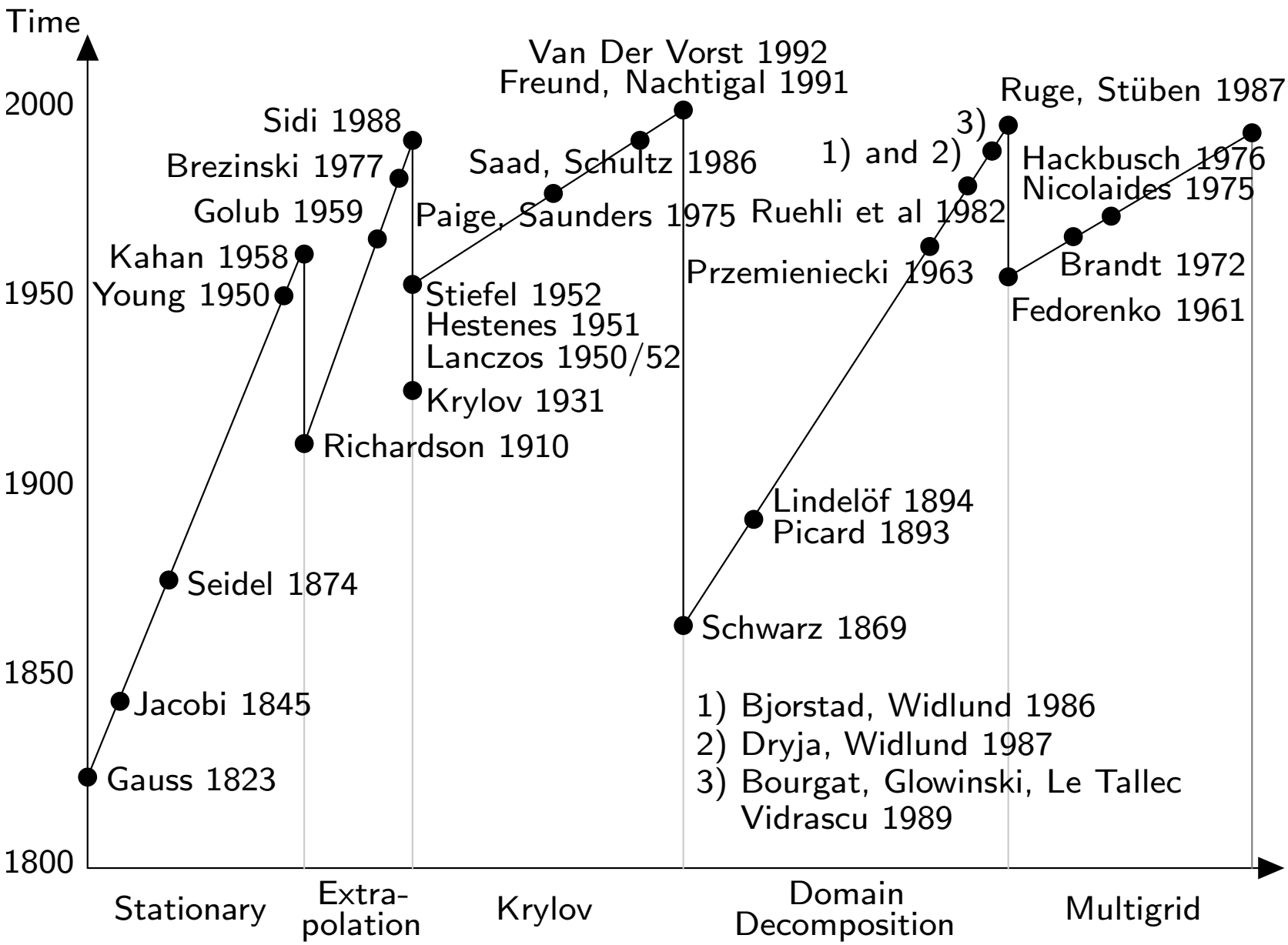
Stationary Methods
Gauss
Jacobi, Seidel
Example
Modern Notation
SOR
Extrapolation
Richardson
Example
Young, Golub, Bresinski, Sidi
Krylov Methods
Conjugate Gradients
Example
Paige, Saunders, Saad, Freund, Van Der Vorst
Domain Decomposition
Invention of Schwarz
Schwarz example
Further DD Methods
Example
Optimal Schwarz
Multigrid
MG example
Fedorenko
Preconditioning

Story-line: Iterative Methods for Linear Systems



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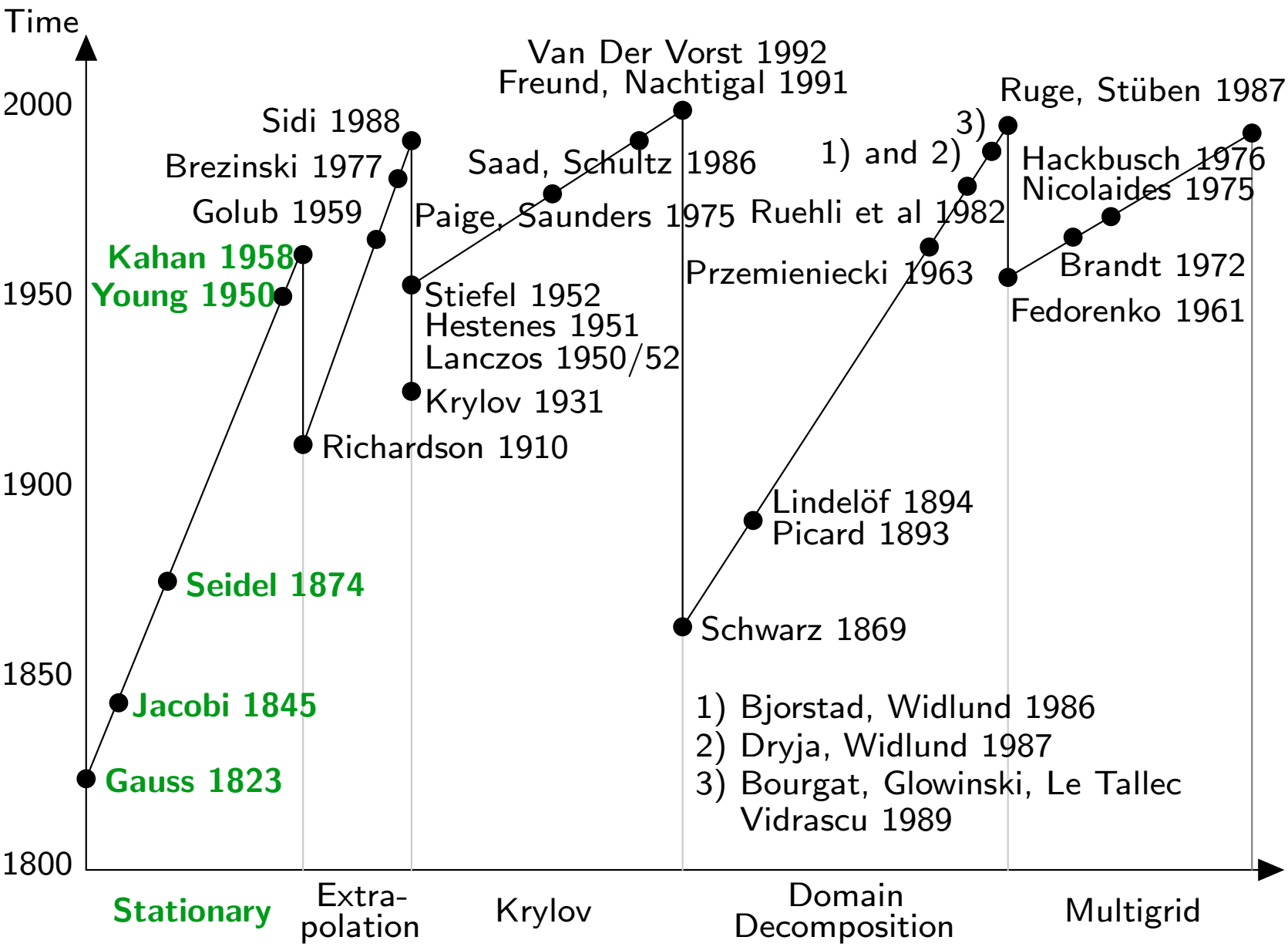
Stationary Iterative Methods



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Gauss Invents an Iterative Method in a Letter

Gauss (1823), in a letter to Gerling: in order to compute a least squares solution based on angle measurements between the locations Berger Warte, Johannisberg, Taufstein and Milseburg:

Die Bedingungsgleichungen sind also:

$$
\begin{aligned}
0 &= +\quad 6 + 67a - 13b - 28c - 26d \\
0 &= -\ 7558 - 13a + 69b - 50c - 6d \\
0 &= -14604 - 28a - 50b + 156c - 78d \\
0 &= +22156 - 26a - 6b - 78c + 110d;
\end{aligned}
$$

$$\text{Summe} = 0.$$

Um nun indirect zu eliminiren, bemerke ich, dass, wenn 3 der Grössen a, b, c, d gleich 0 gesetzt werden, die vierte den grössten Werth bekommt, wenn d dafür gewählt wird. Natürlich muss jede Grösse aus ihrer eigenen Gleichung, also d aus der vierten, bestimmt werden. Ich setze also $d = -201$ und substituire diesen Werth. Die absoluten Theile werden dann: $+5232$, -6352, $+1074$, $+46$; das Übrige bleibt dasselbe.

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Gauss Concludes his Letter

Fast jeden Abend mache ich eine neue Auflage des Tableaus, wo immer leicht nachzuhelfen ist. Bei der Einförmigkeit des Messungsgeschäfts gibt dies immer eine angenehme Unterhaltung; man sieht dann auch immer gleich, ob etwas zweifelhaftes eingeschlichen ist, was noch wünschenswerth bleibt, etc. Ich empfehle Ihnen diesen Modus zur Nachahmung. Schwerlich werden Sie je wieder direct eliminiren, wenigstens nicht, wenn Sie mehr als 2 Unbekannte haben. Das indirecte Verfahren lässt sich halb im Schlafe ausführen, oder man kann während desselben an andere Dinge denken.

Jacobi (1845): Ueber eine neue Auflösungsart der bei der Methode der kleinsten Quadrate vorkommenden lineären Gleichungen

Seidel (1874): Ueber ein Verfahren, die Gleichungen, auf welche die Methode der kleinsten Quadrate führt, sowie lineäre Gleichungen ueberhaupt, durch successive Annäherung aufzulösen

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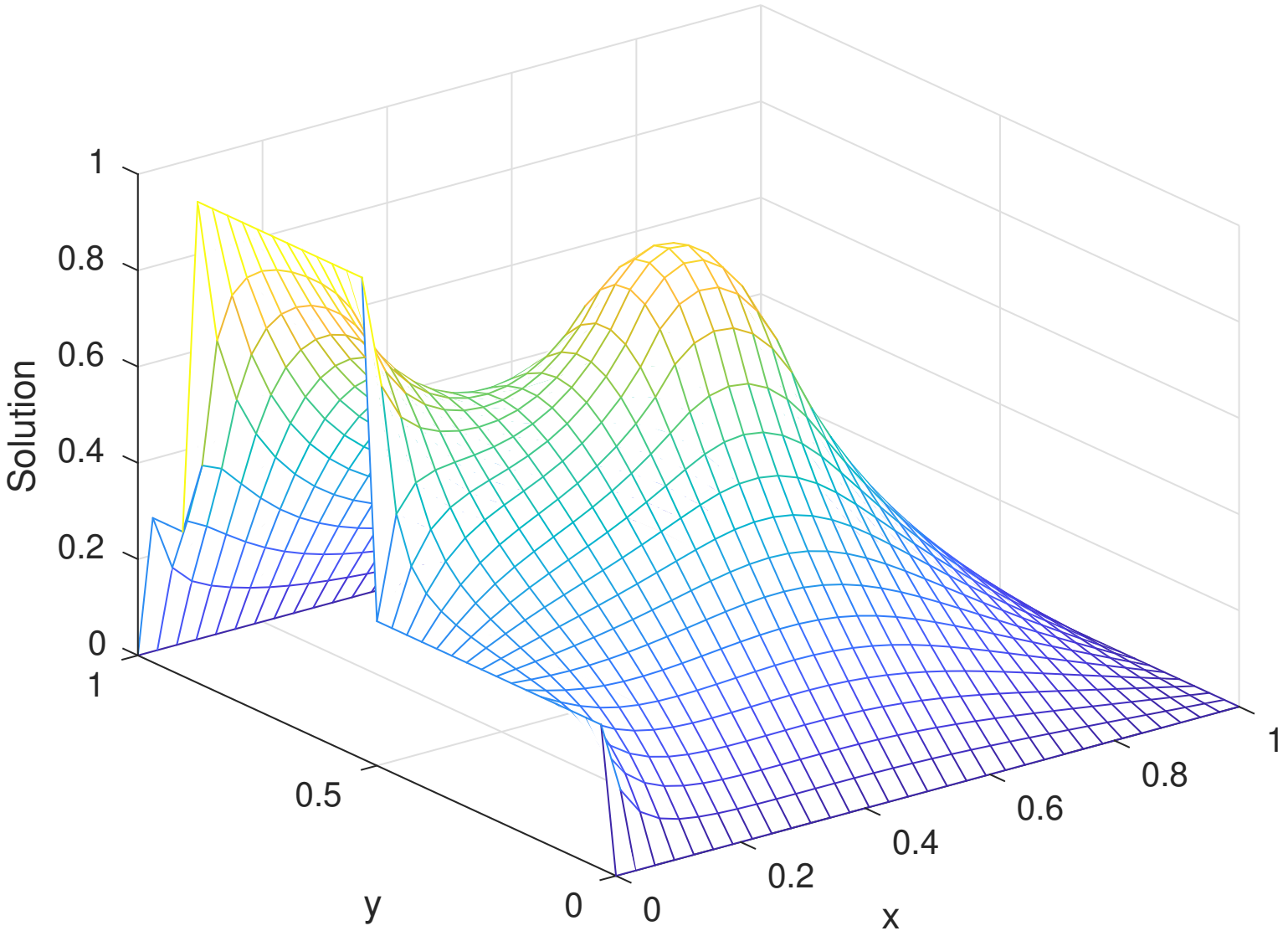
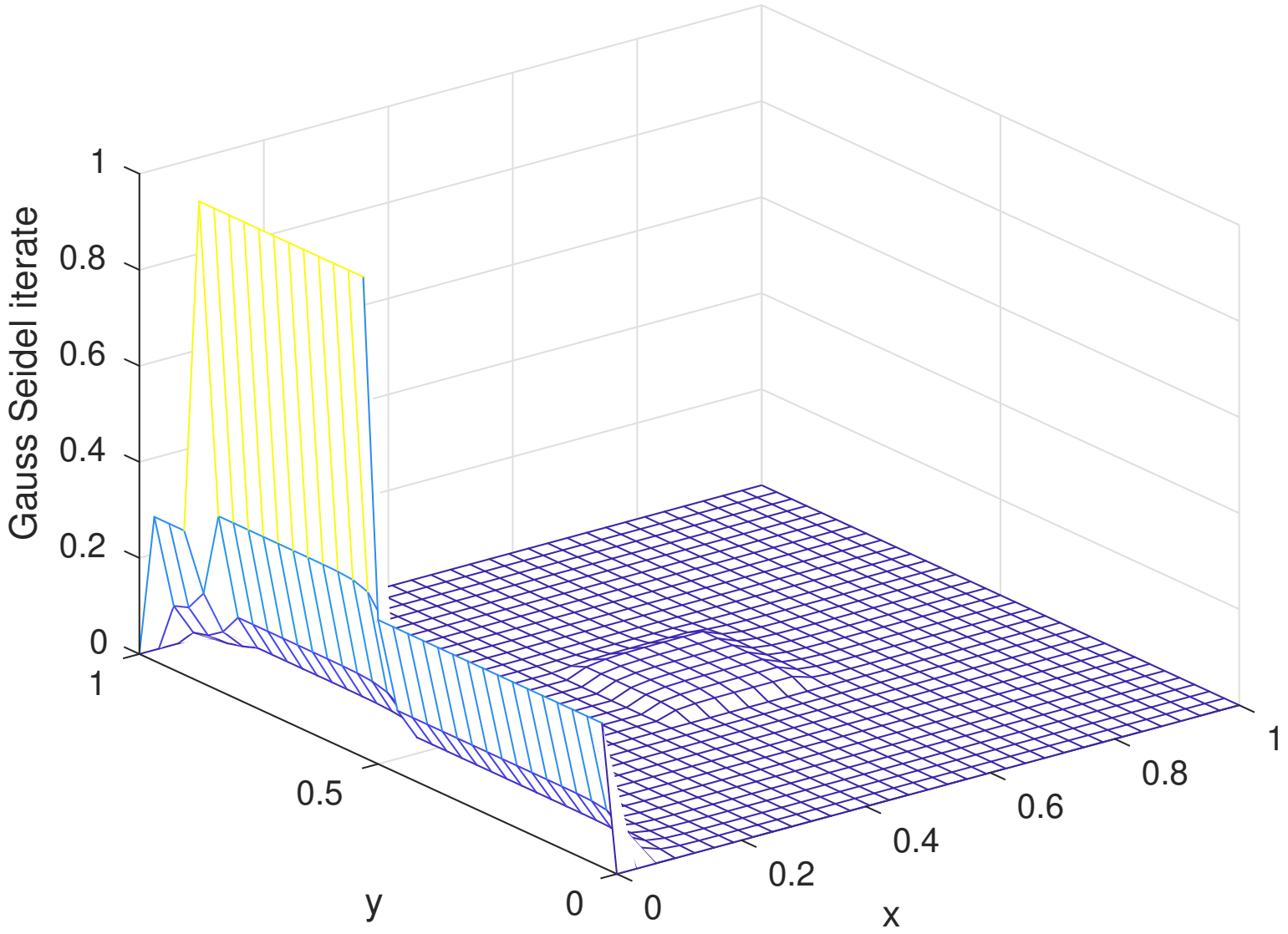
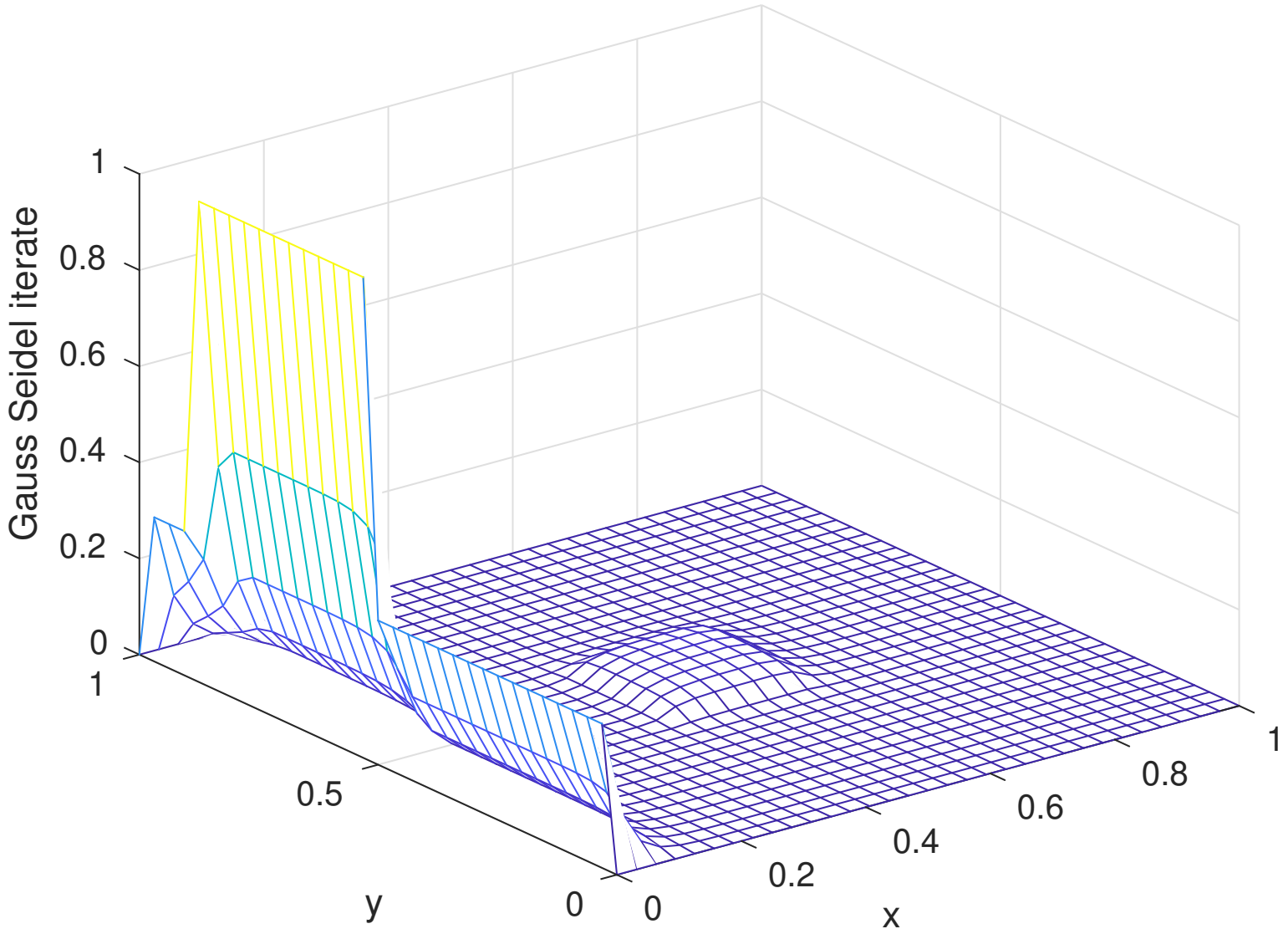
Example: temperature in a room



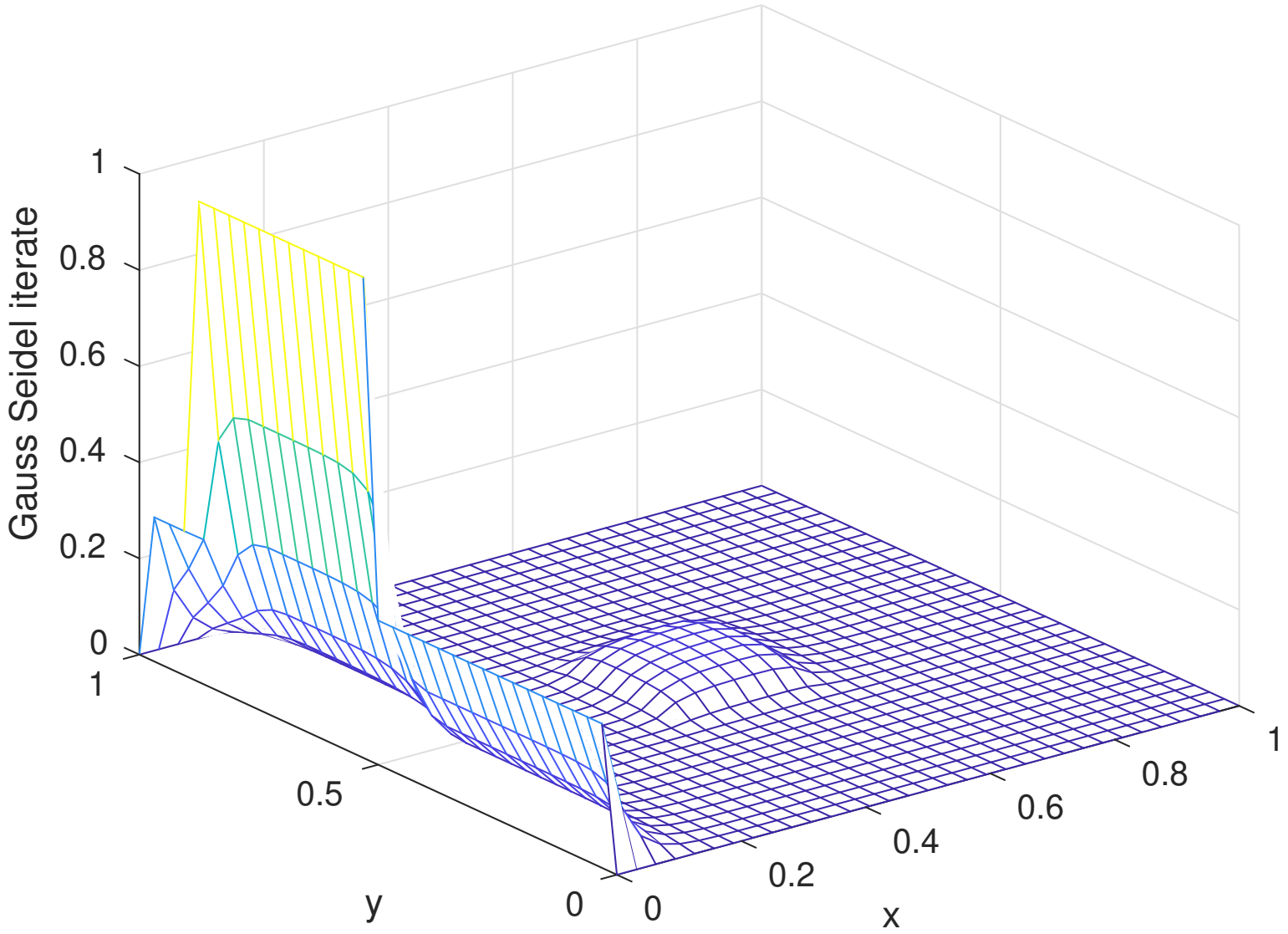
Gauss-Seidel iteration 1



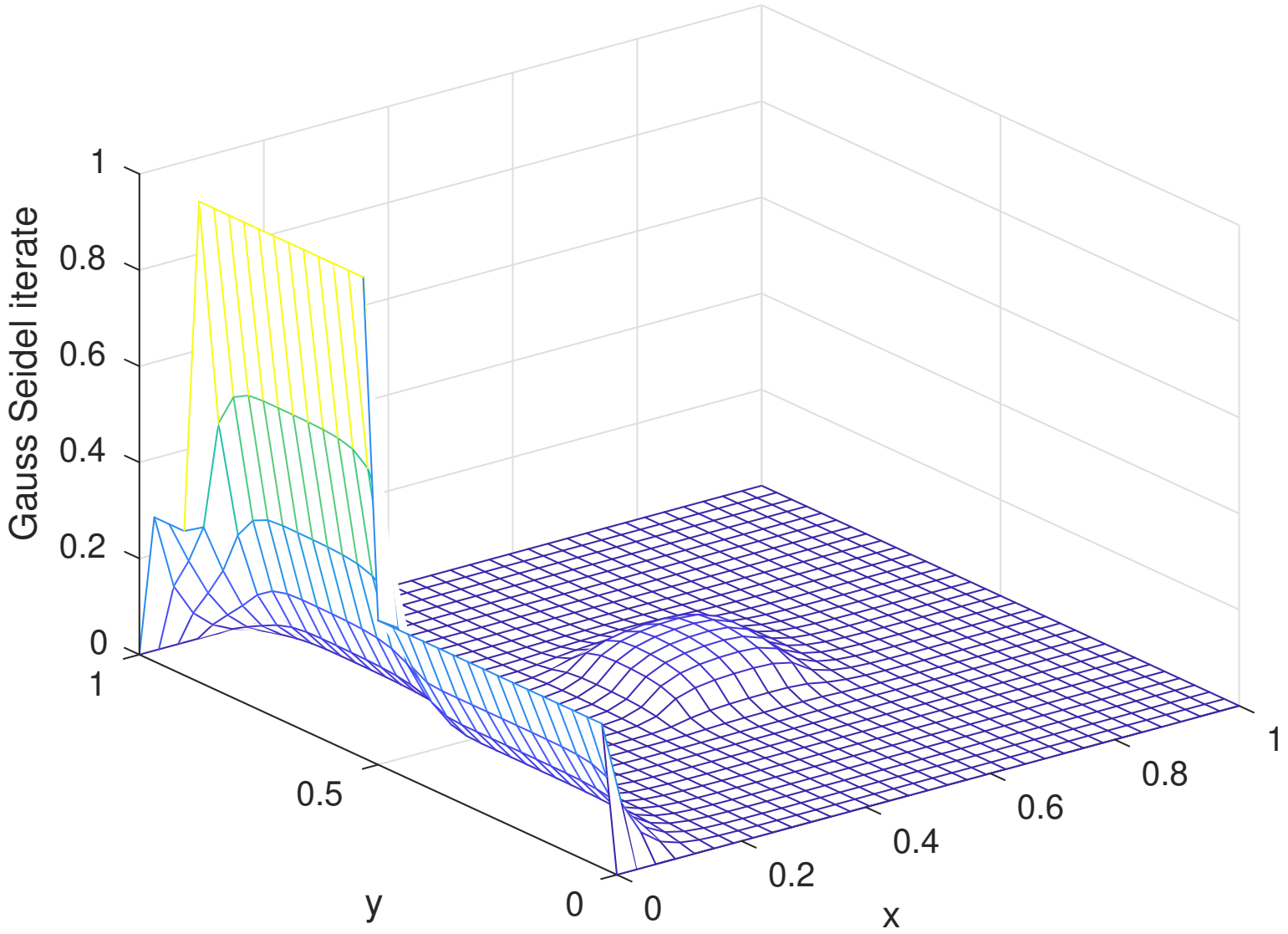
Gauss-Seidel iteration 2



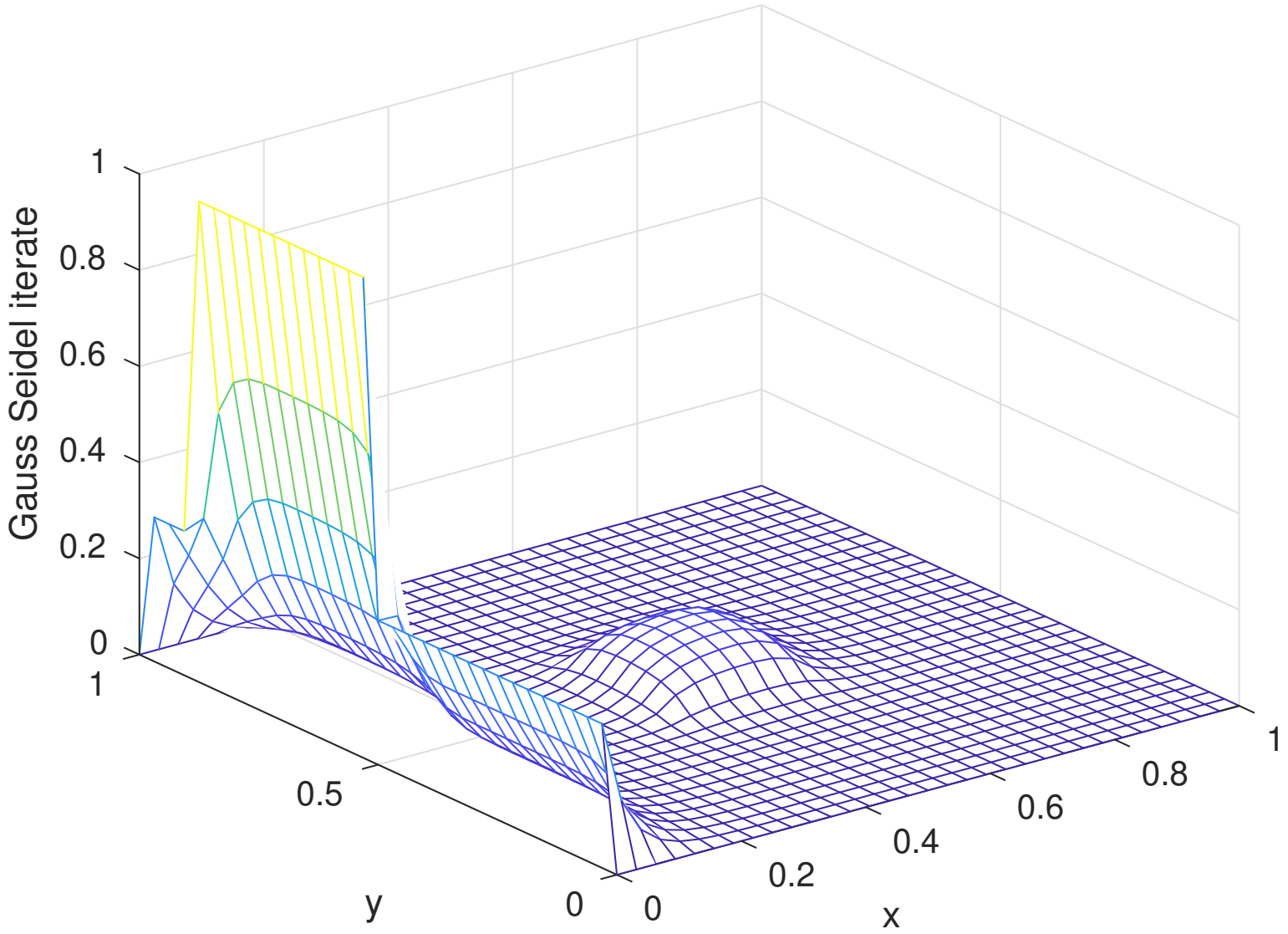
Gauss-Seidel iteration 3



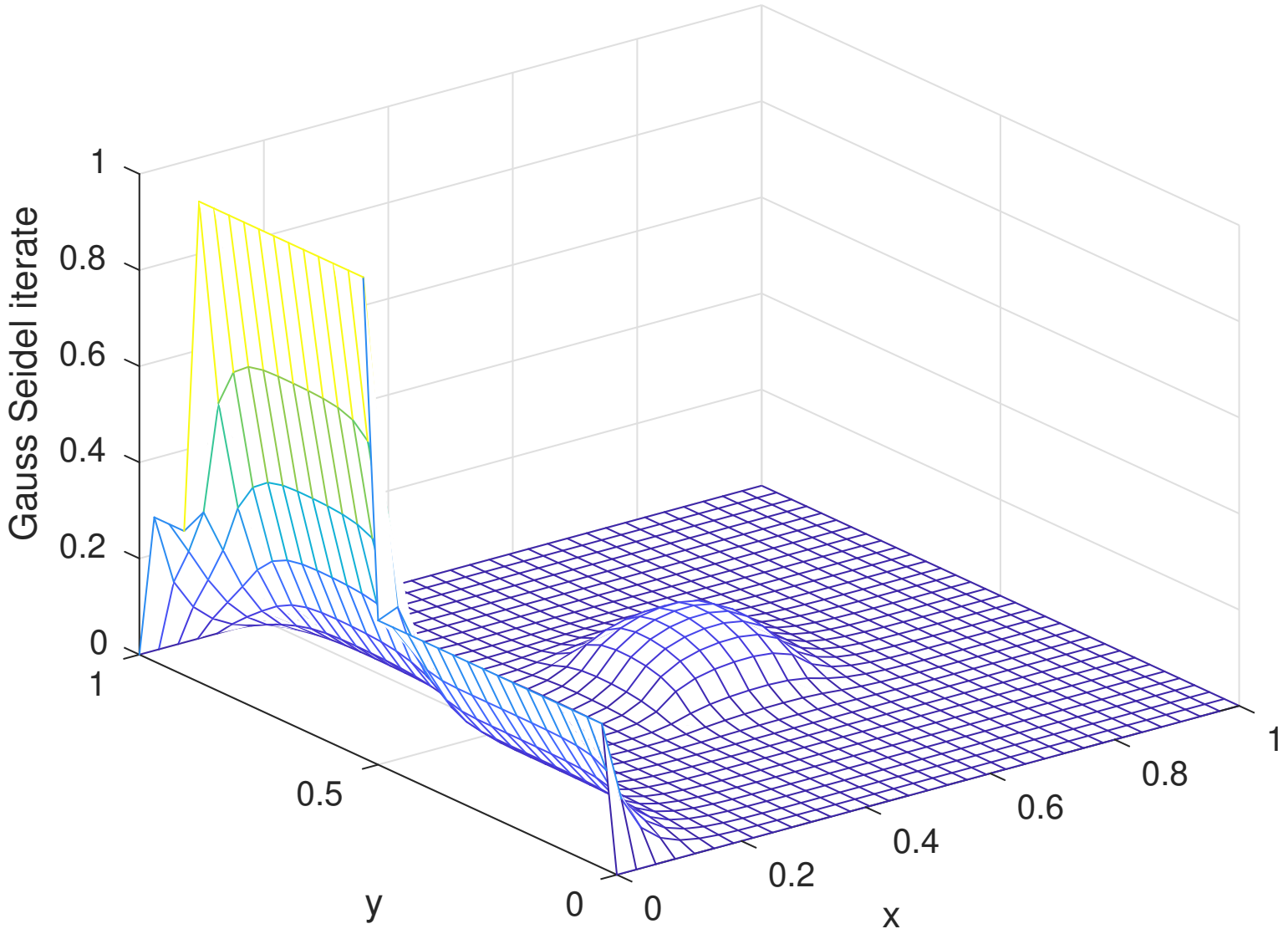
Gauss-Seidel iteration 4



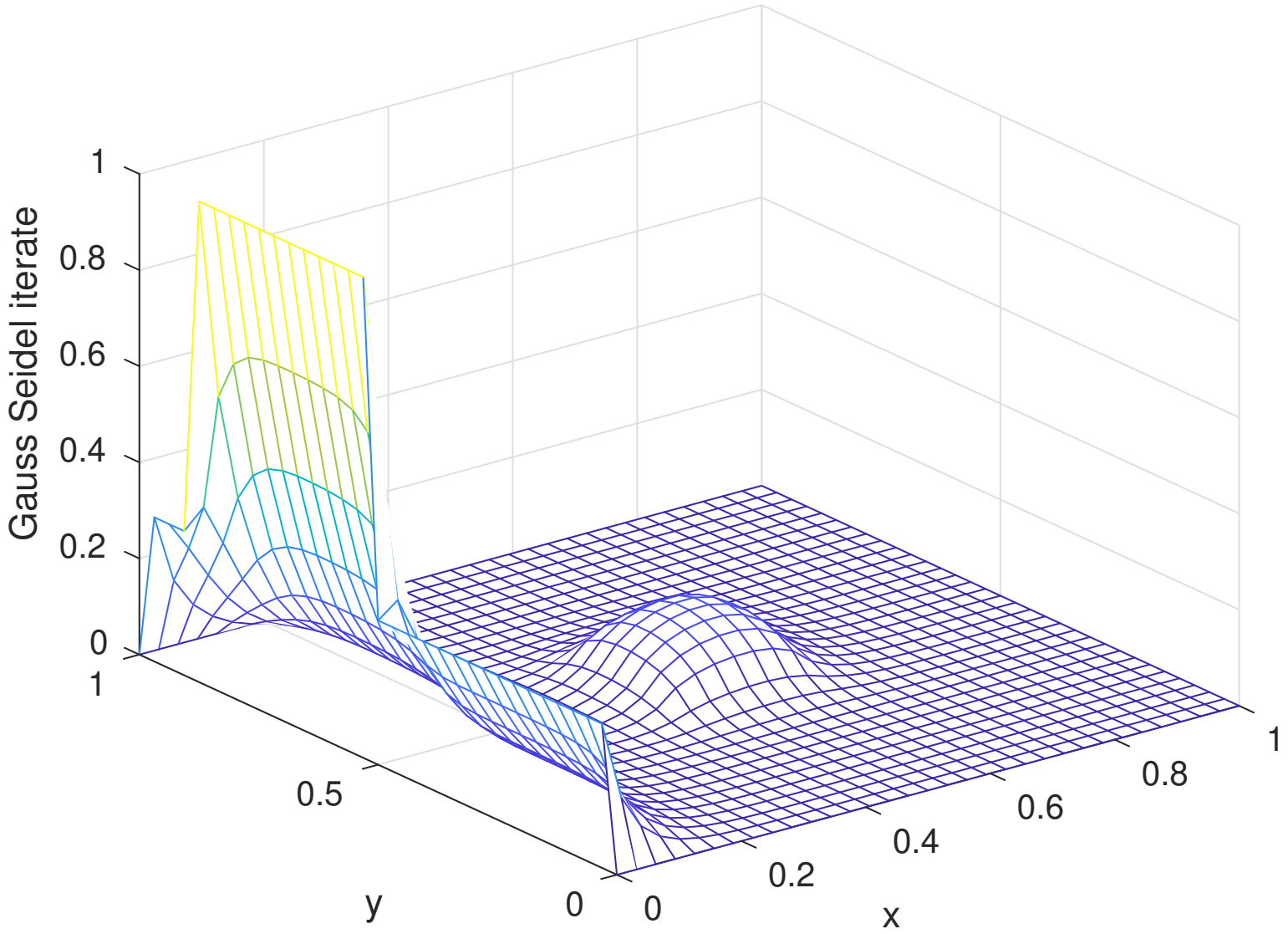
Gauss-Seidel iteration 5



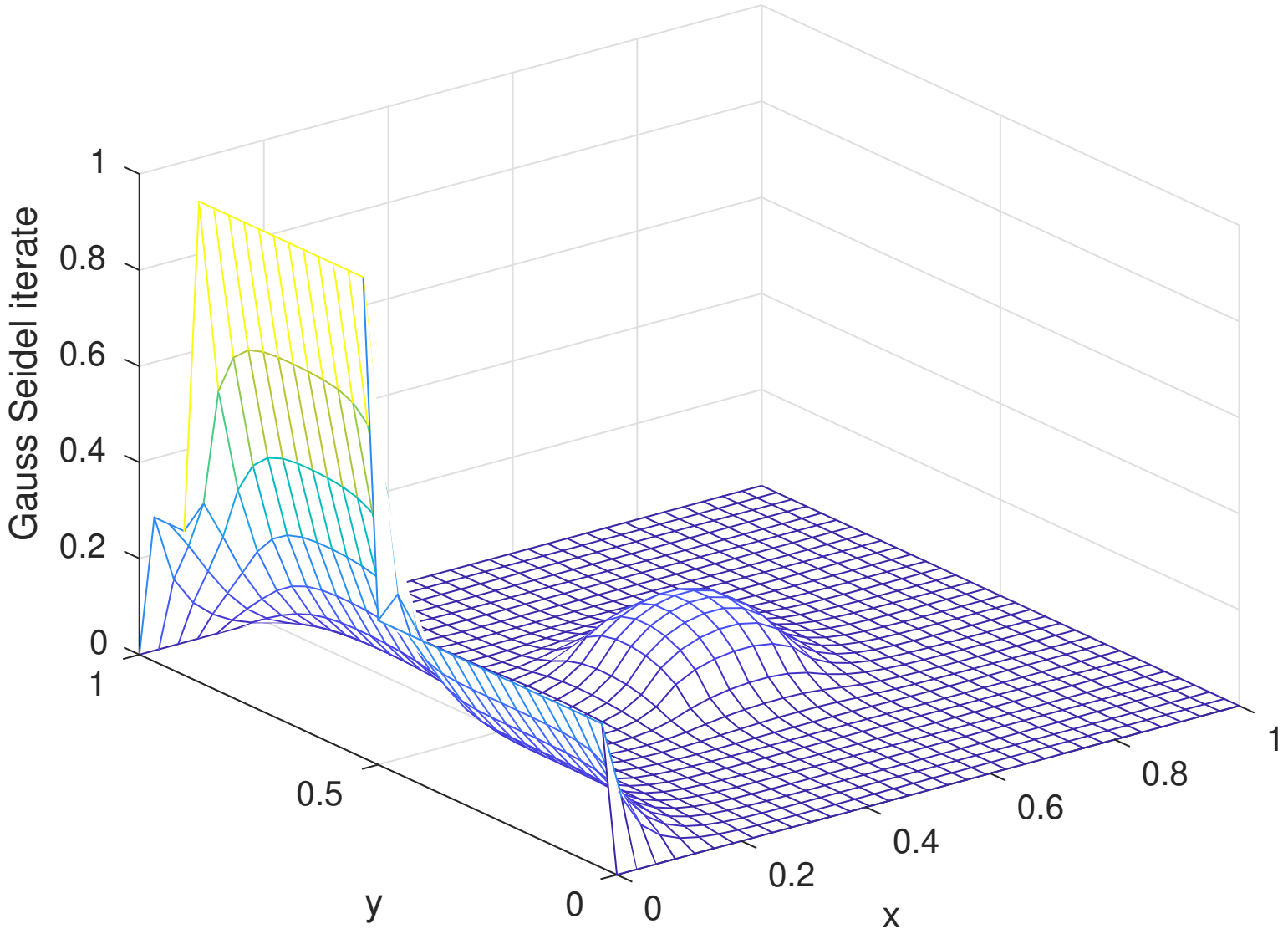
Gauss-Seidel iteration 6



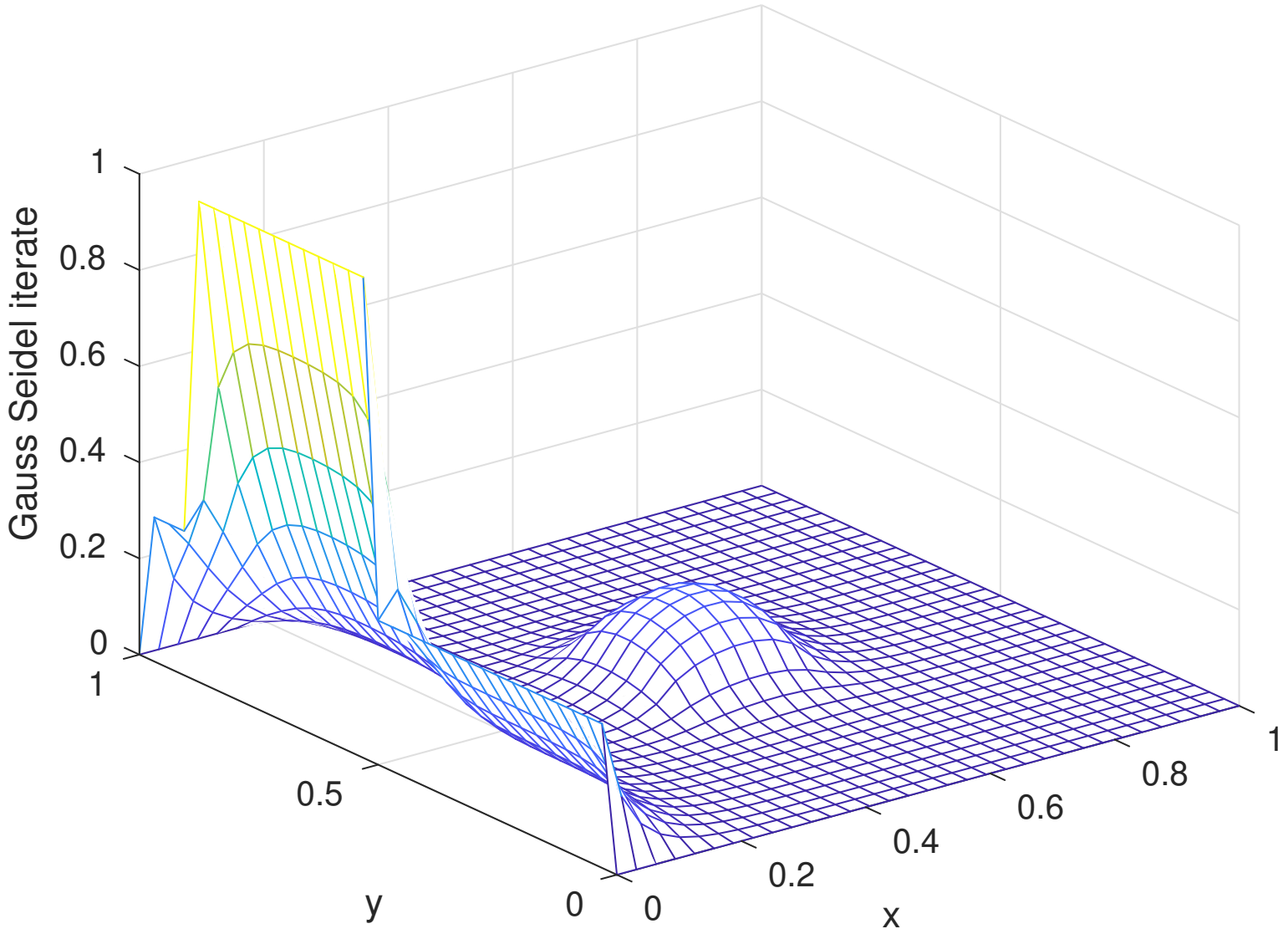
Gauss-Seidel iteration 7



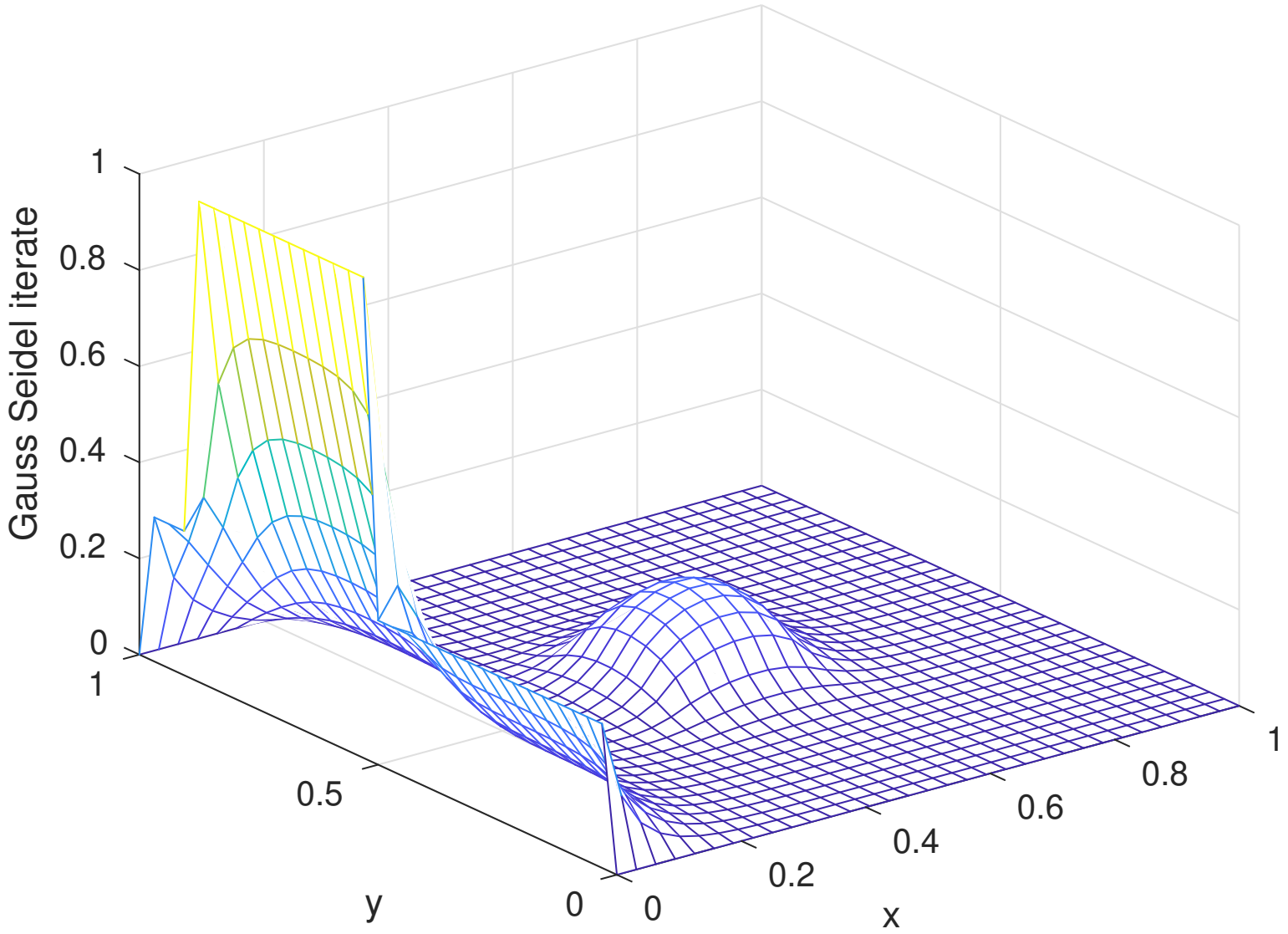
Gauss-Seidel iteration 8



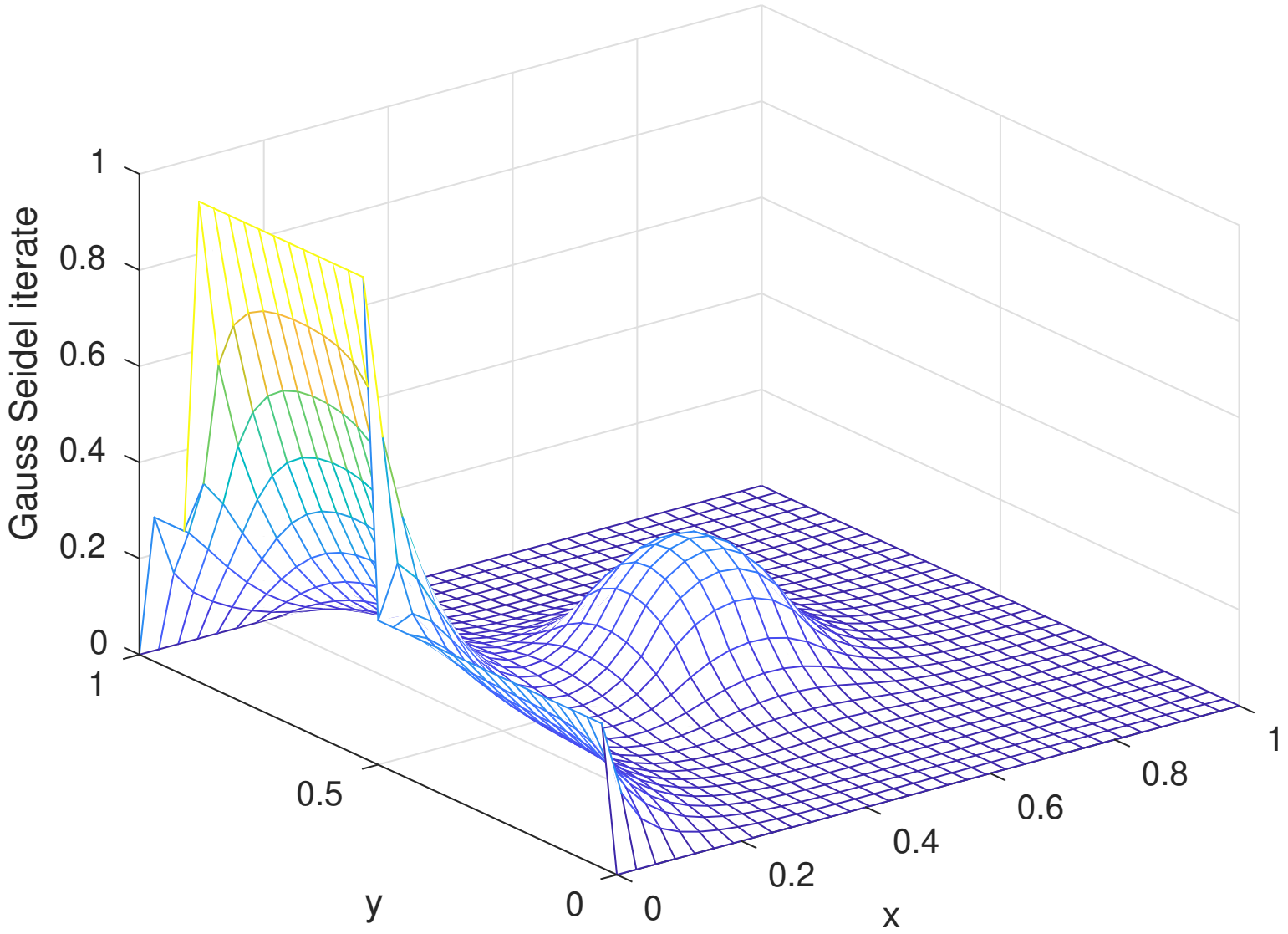
Gauss-Seidel iteration 9



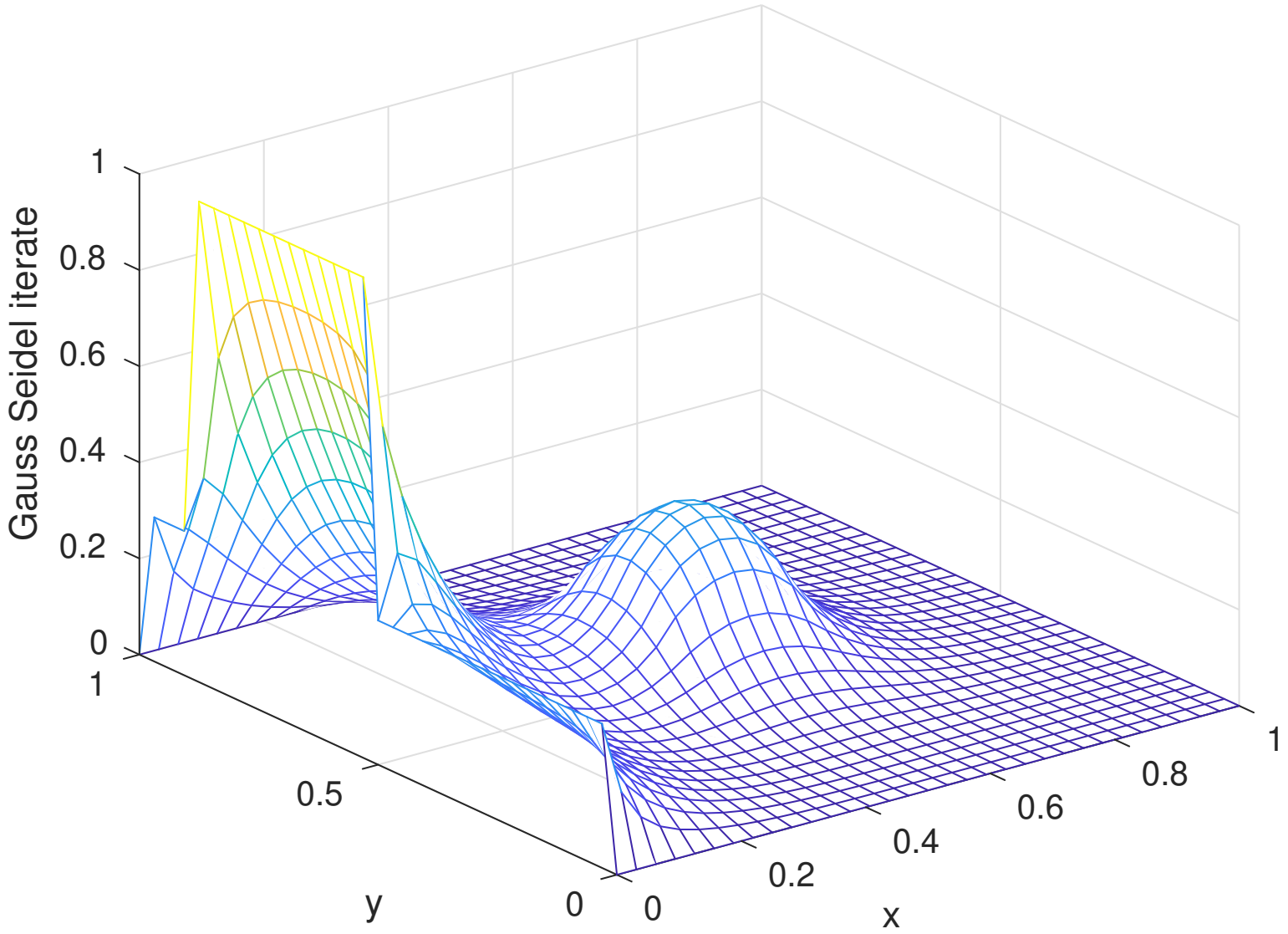
Gauss-Seidel iteration 10



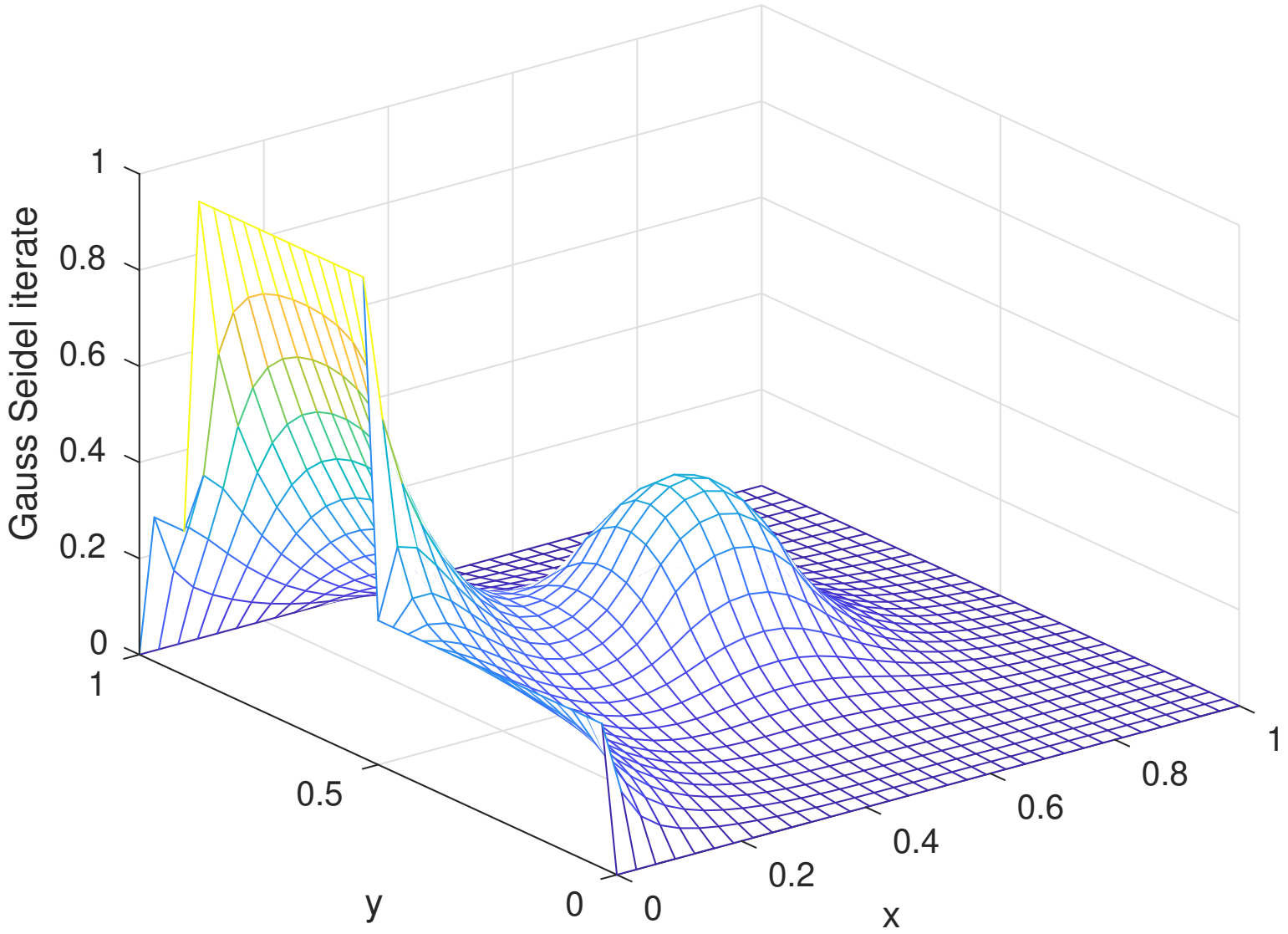
Gauss-Seidel iteration 20



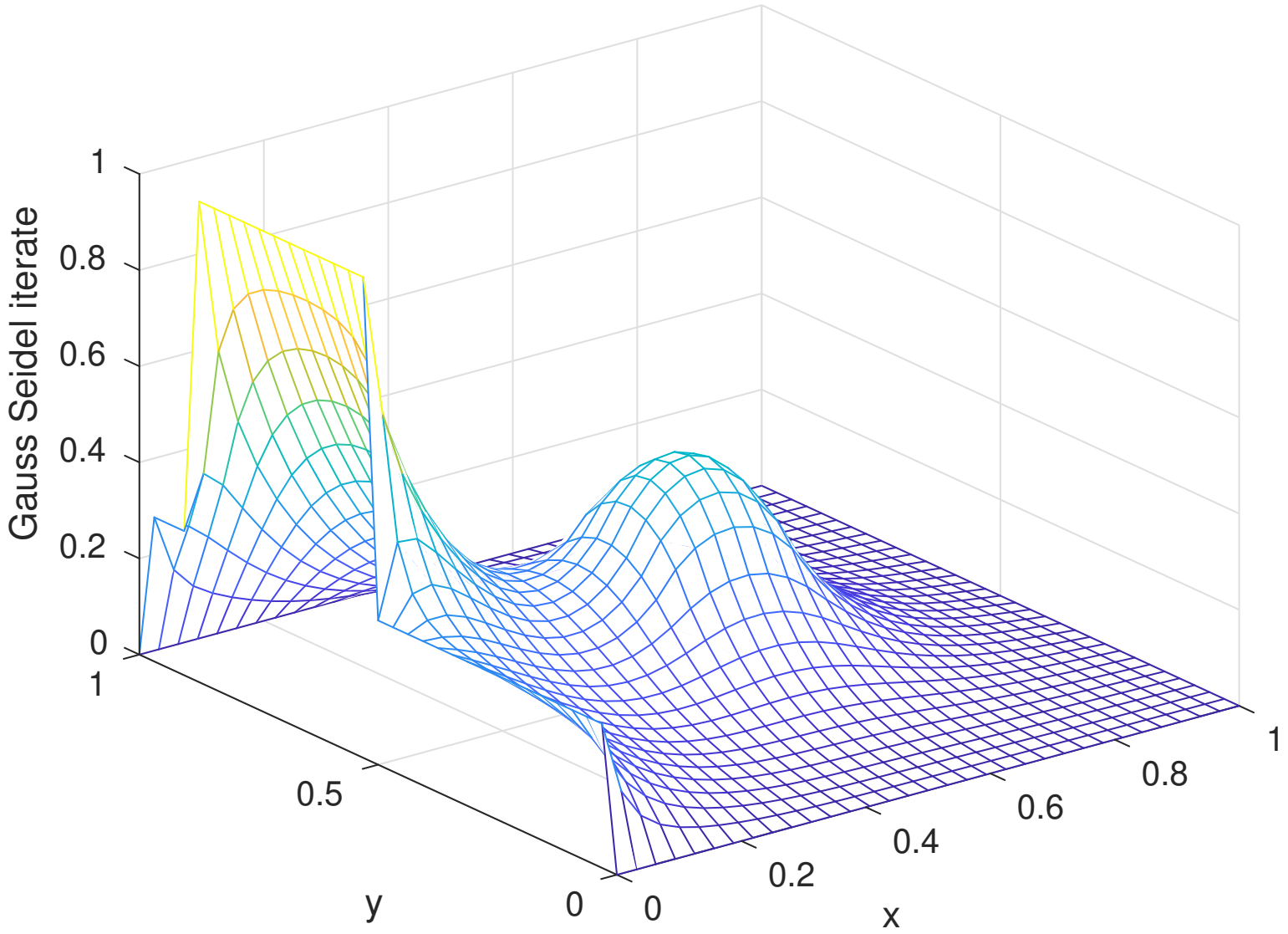
Gauss-Seidel iteration 30



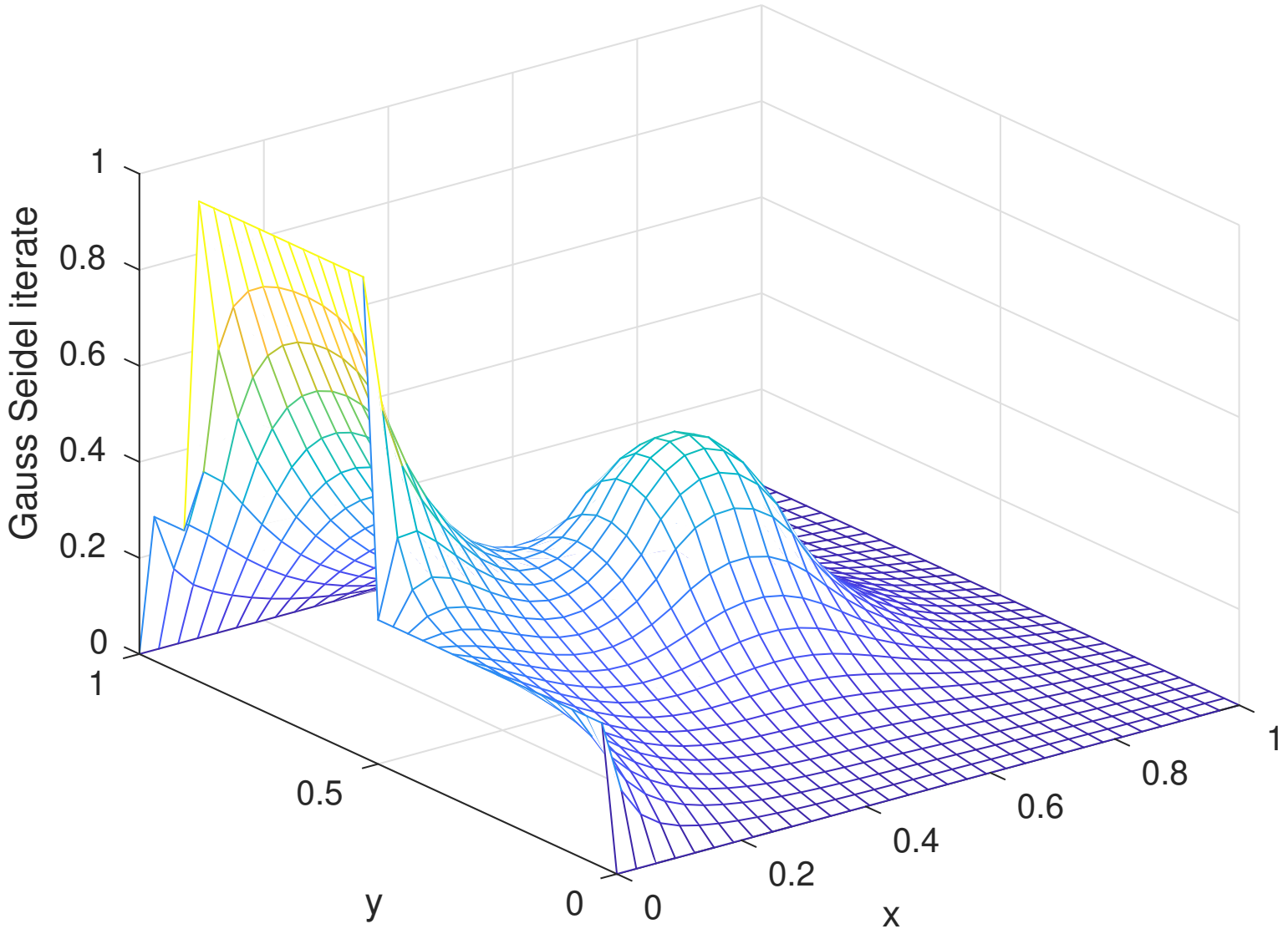
Gauss-Seidel iteration 40



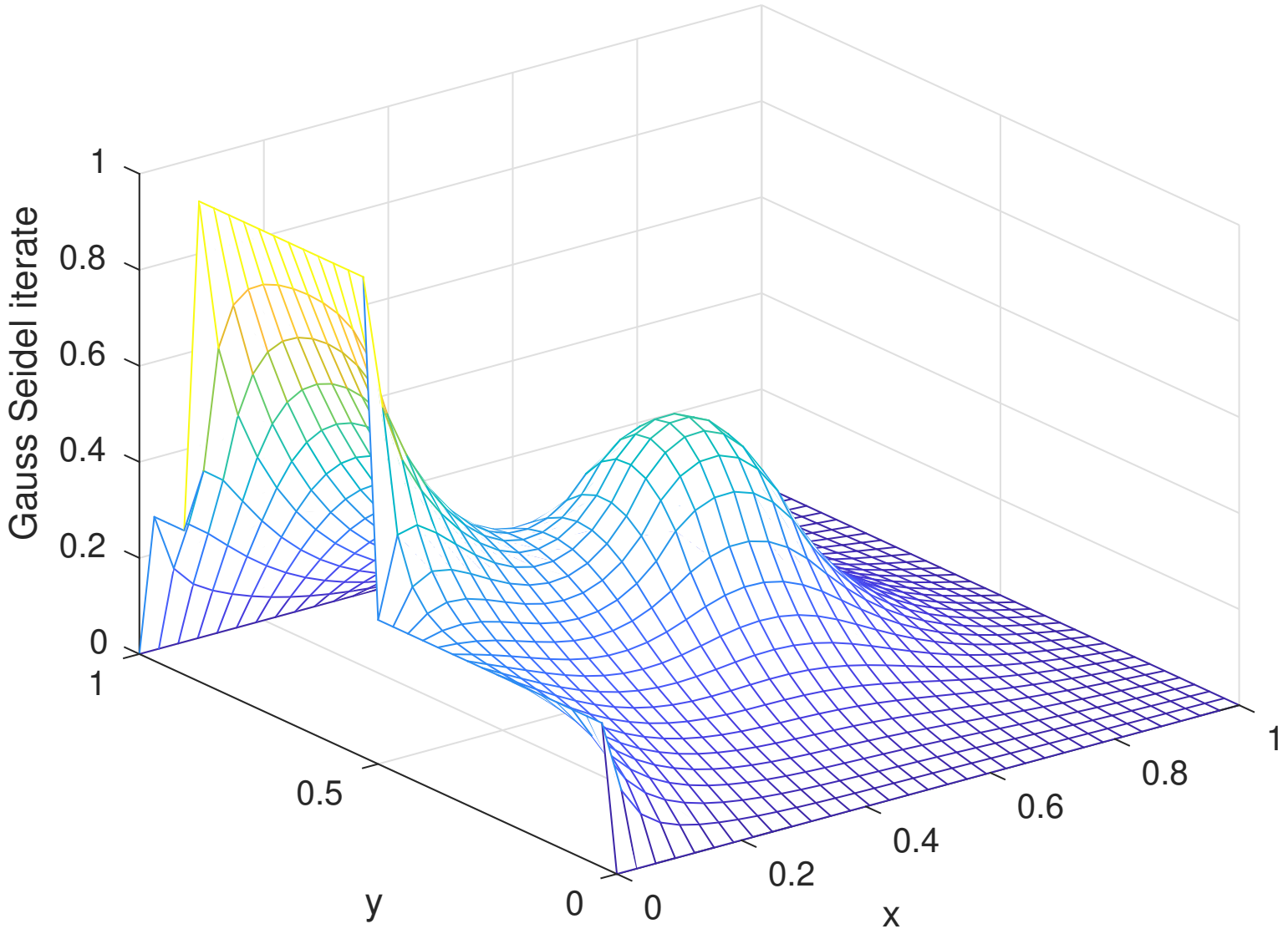
Gauss-Seidel iteration 50



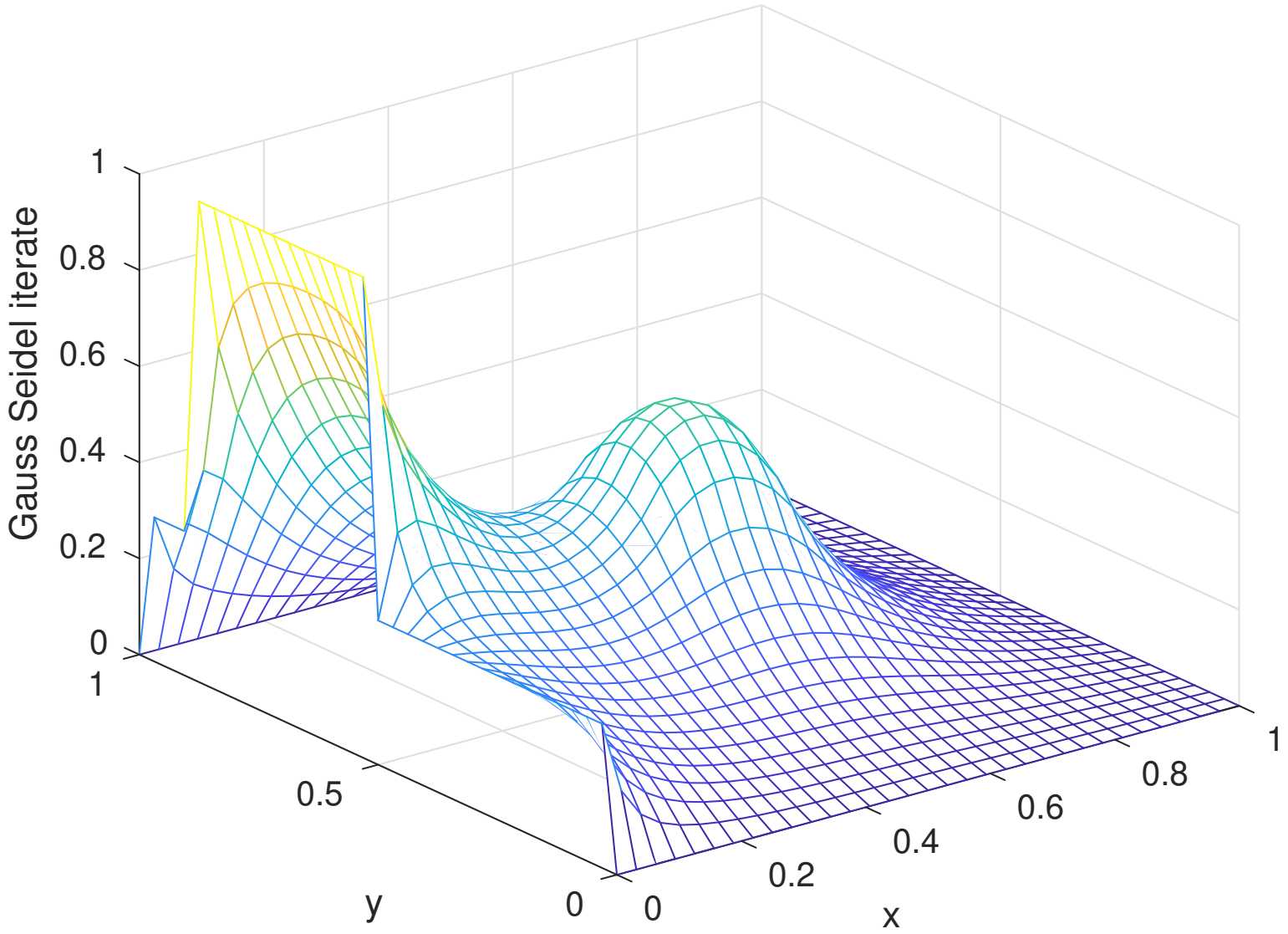
Gauss-Seidel iteration 60



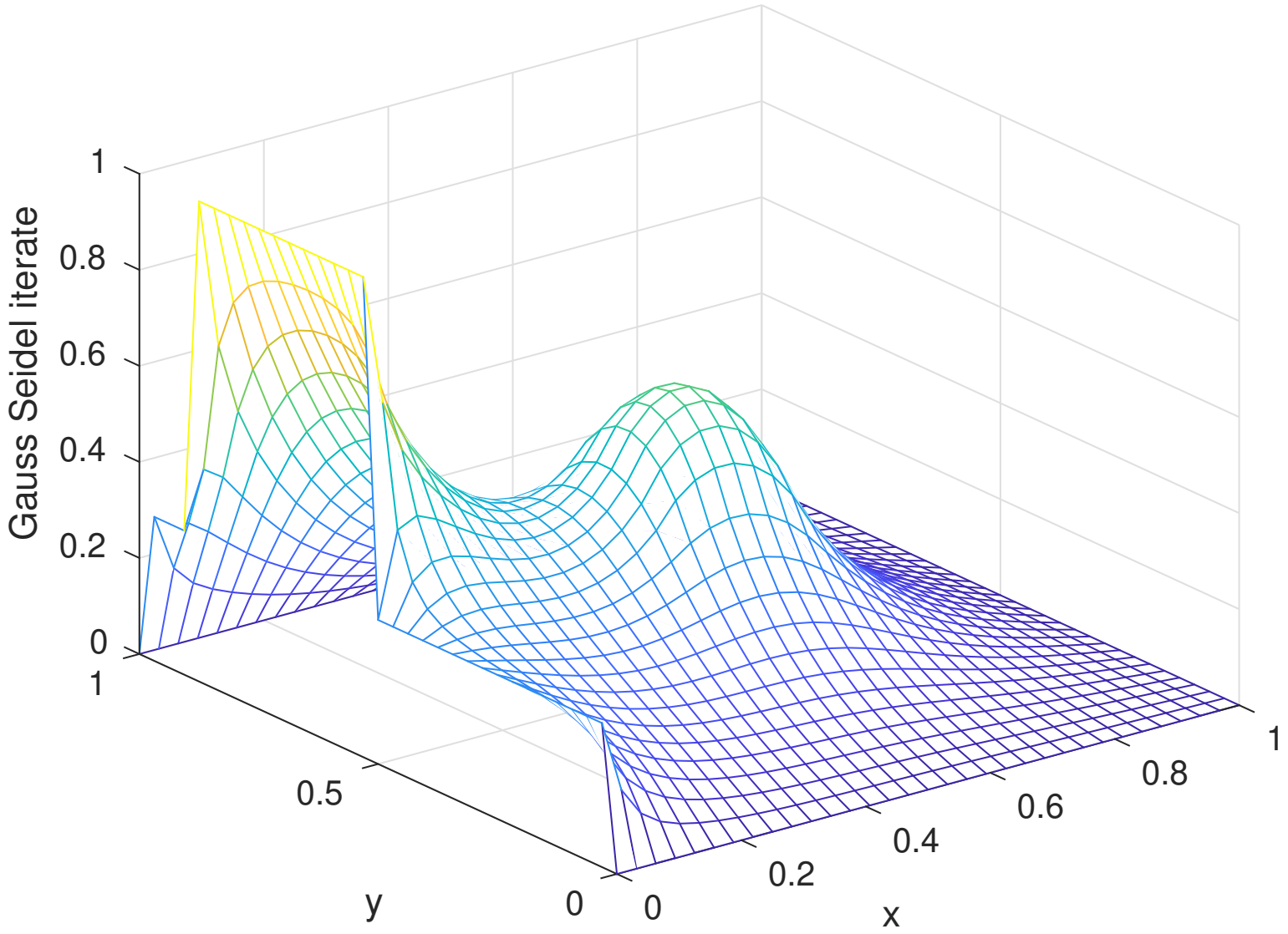
Gauss-Seidel iteration 70



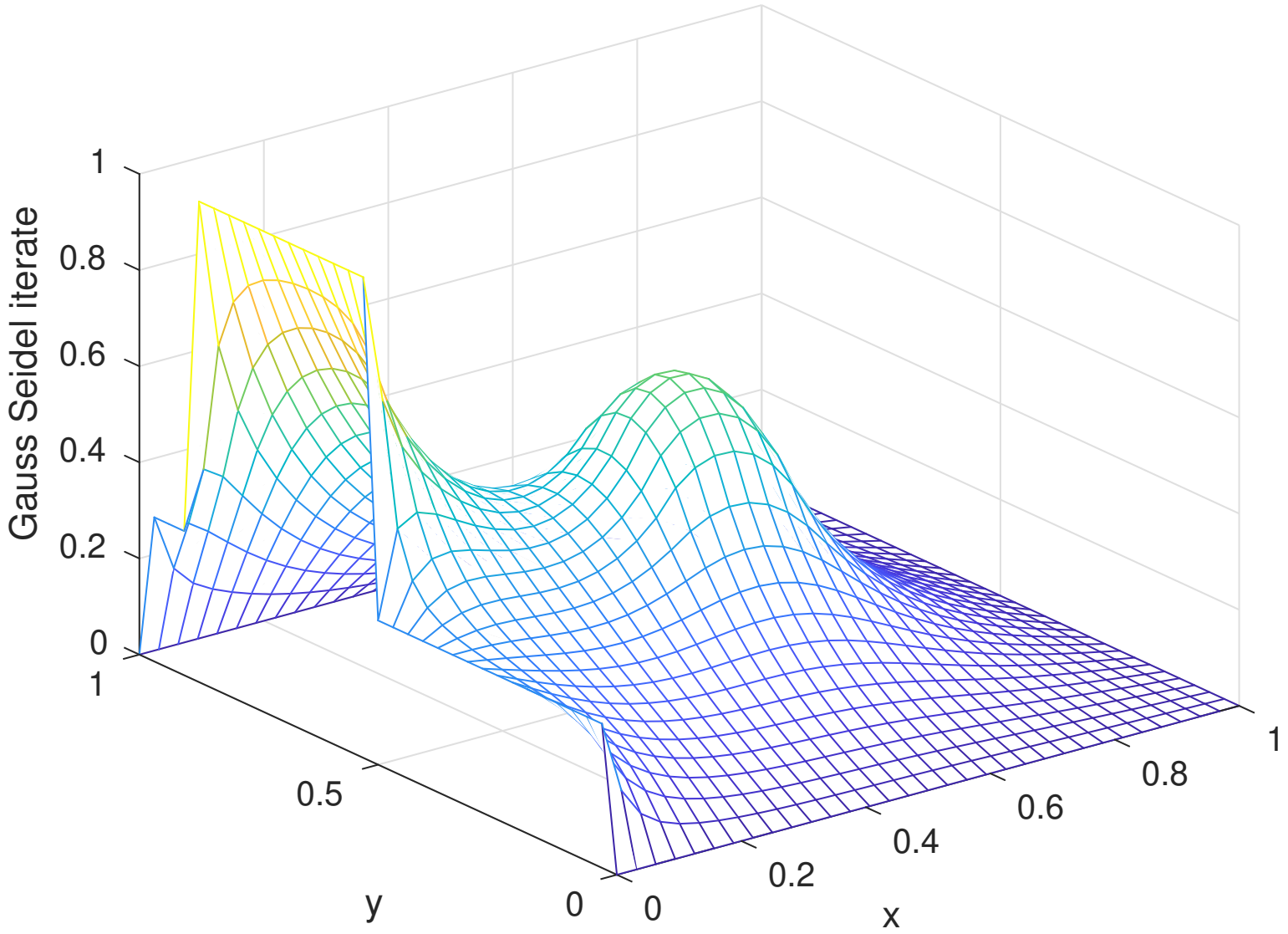
Gauss-Seidel iteration 80



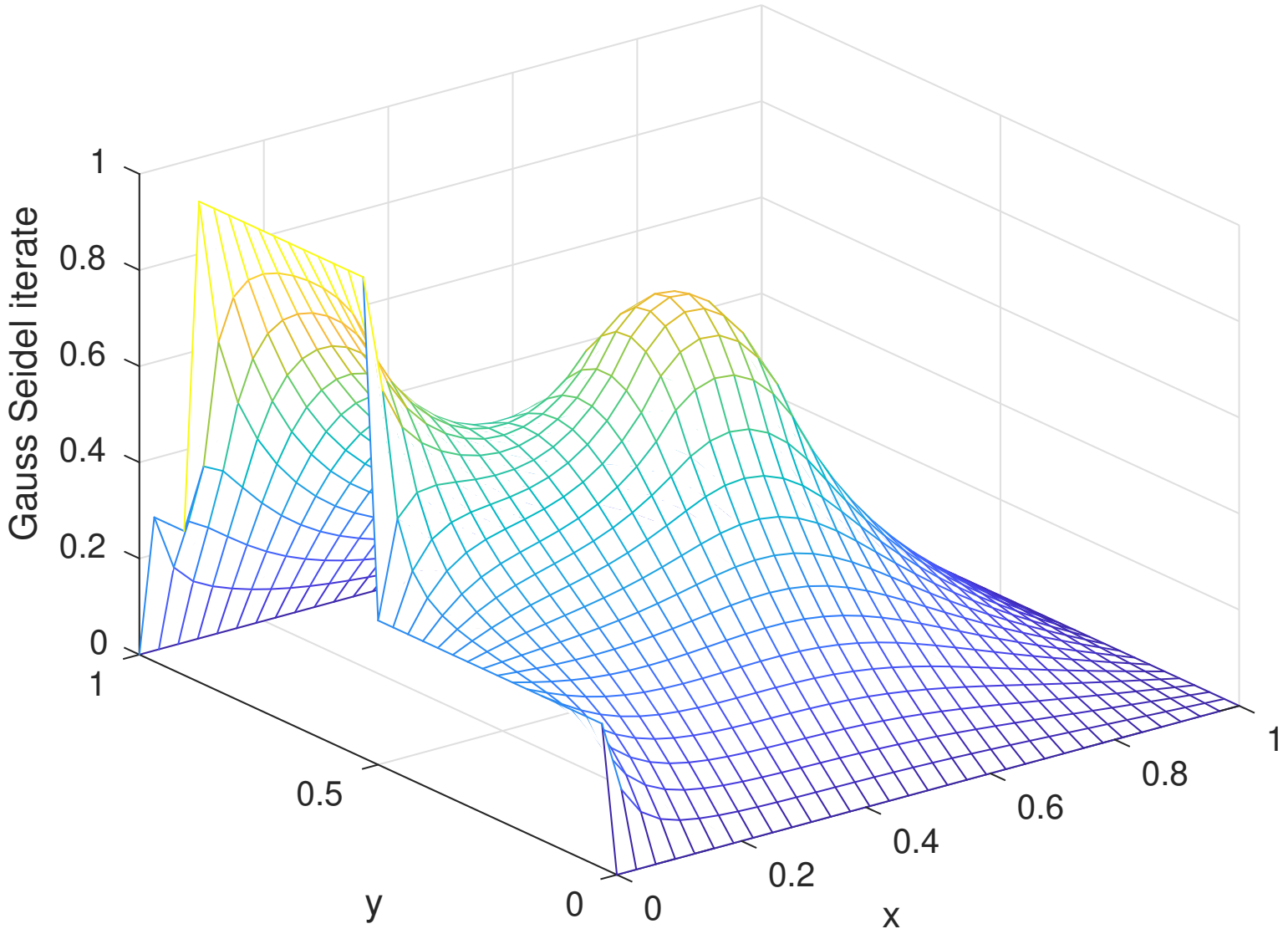
Gauss-Seidel iteration 90



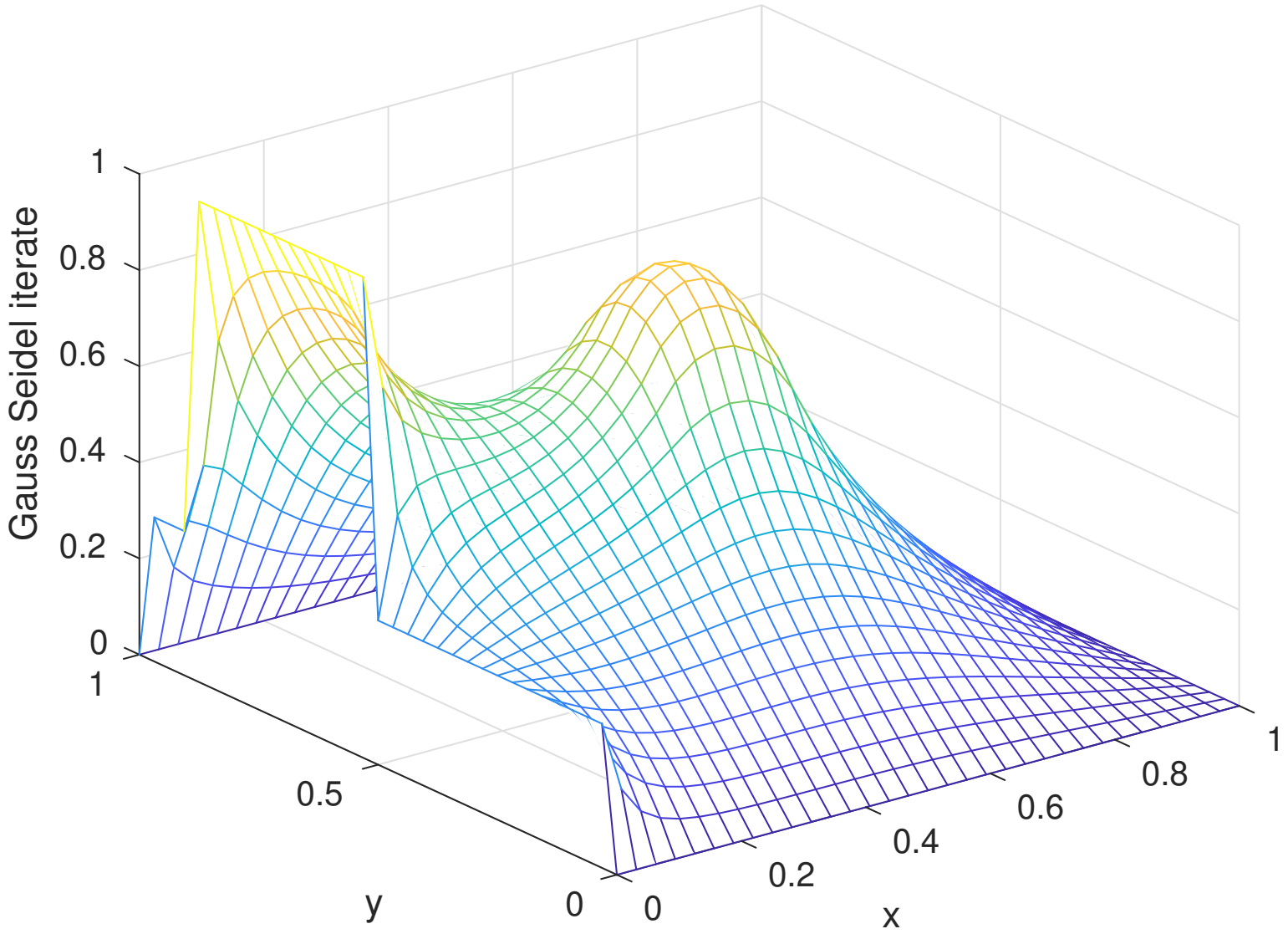
Gauss-Seidel iteration 100



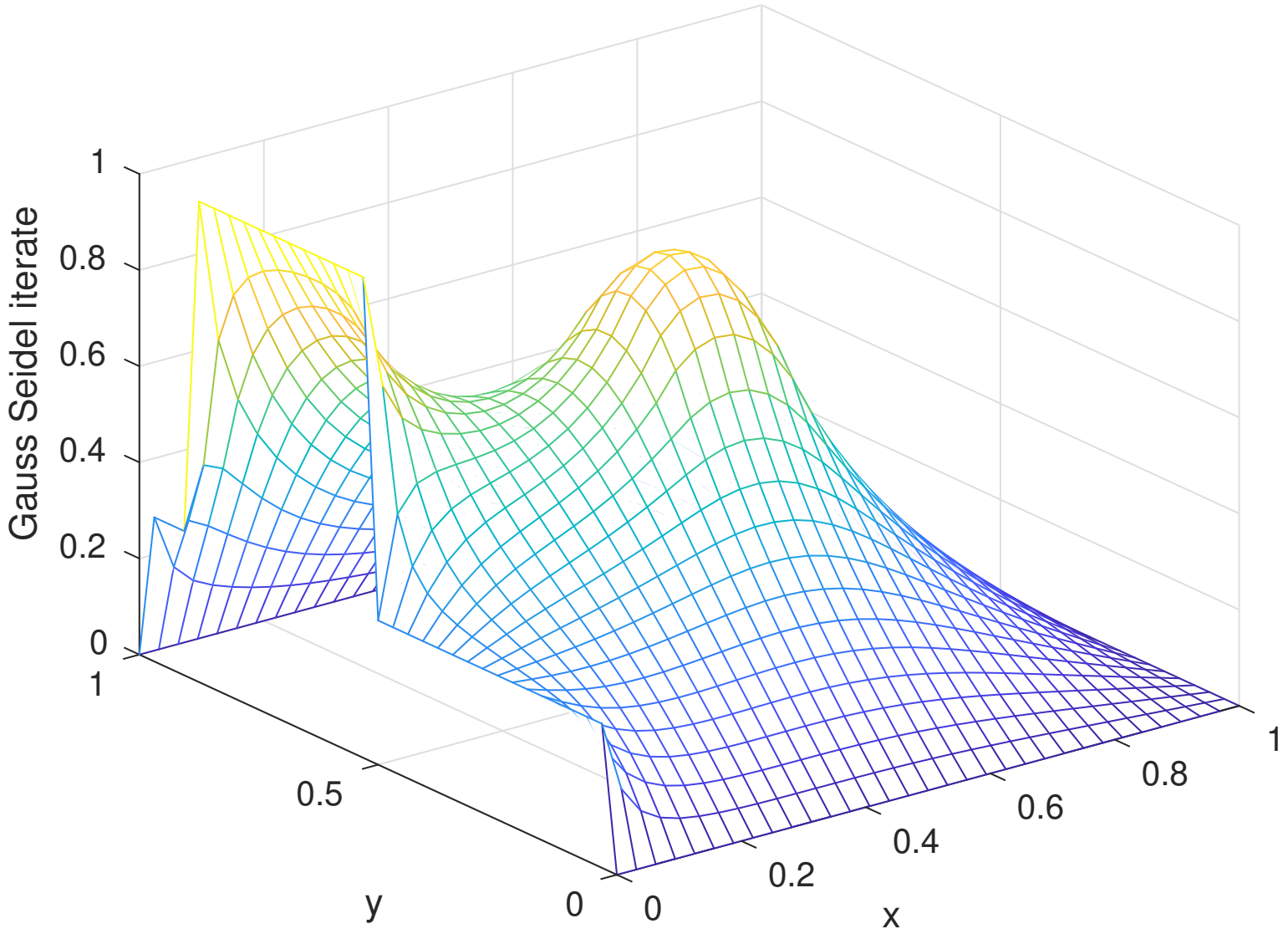
Gauss-Seidel iteration 200



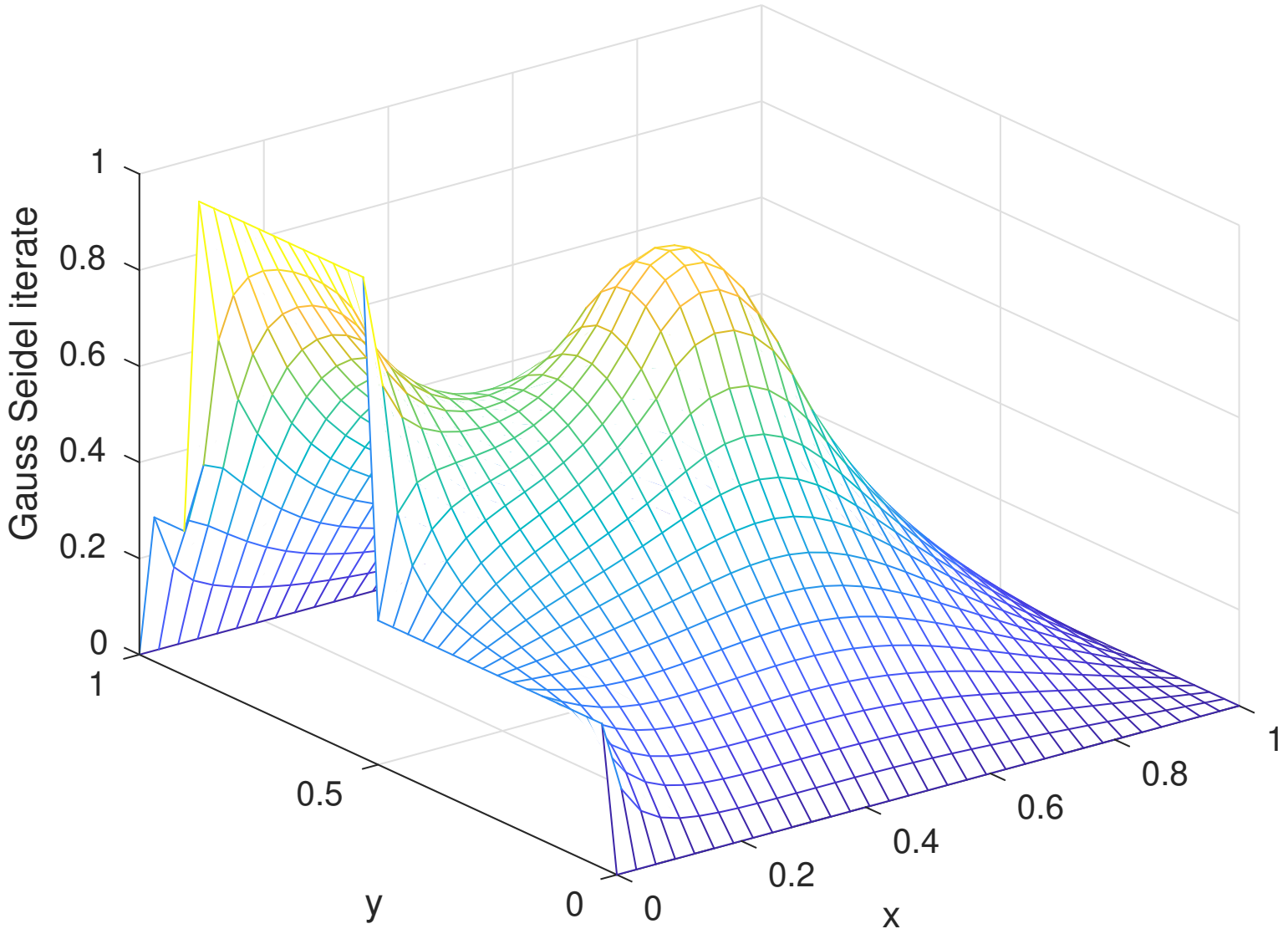
Gauss-Seidel iteration 300



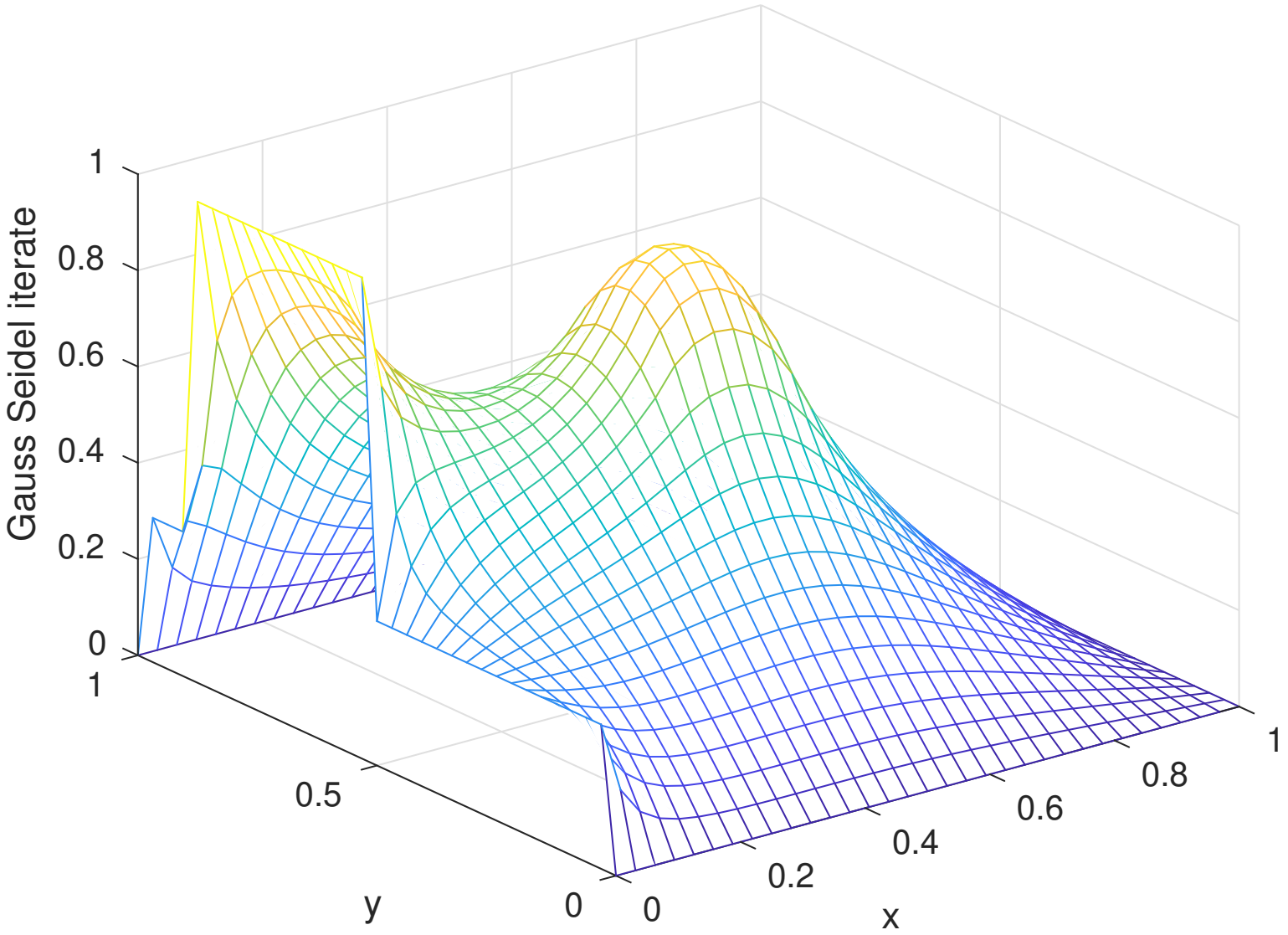
Gauss-Seidel iteration 400



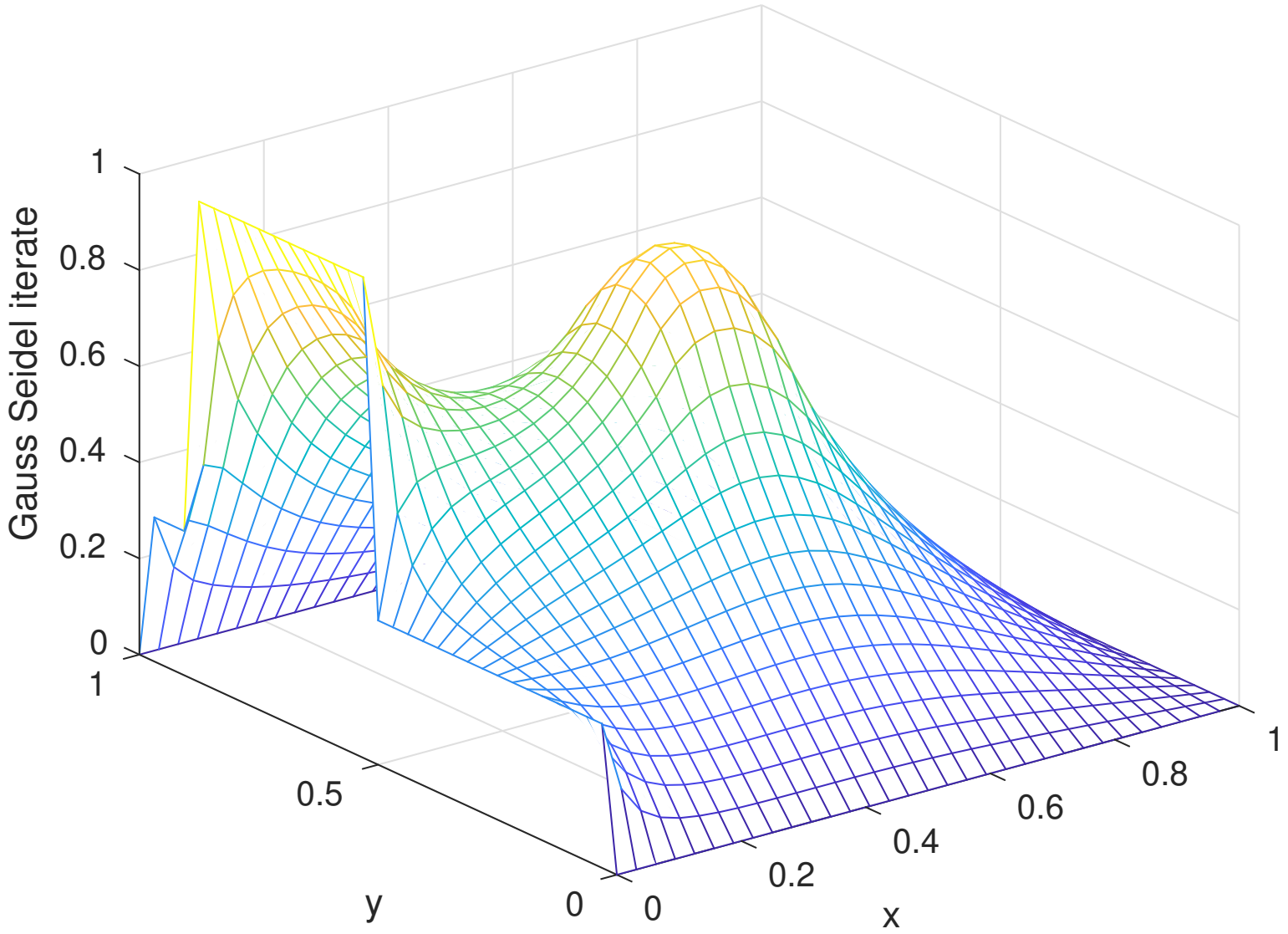
Gauss-Seidel iteration 500



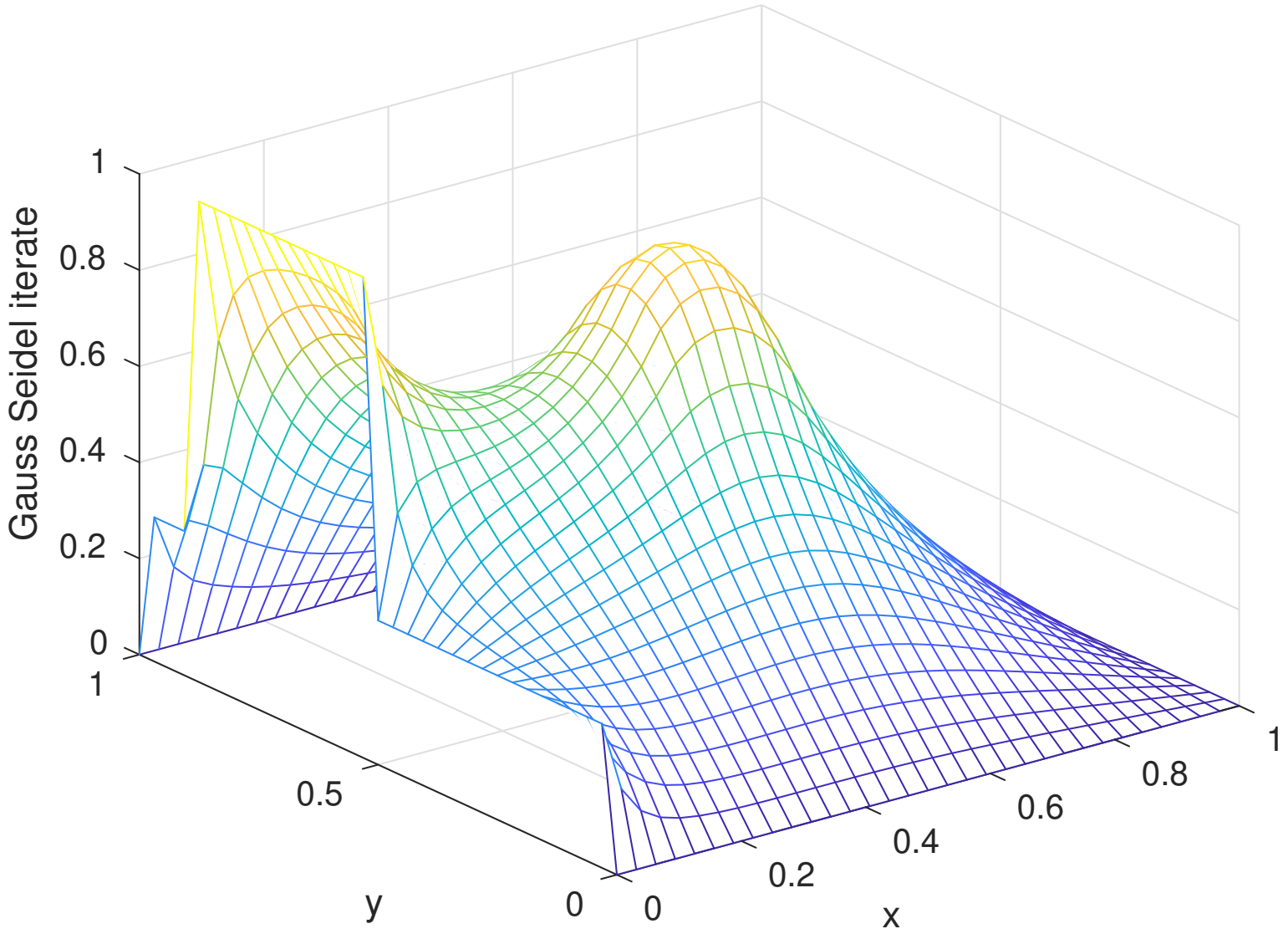
Gauss-Seidel iteration 600



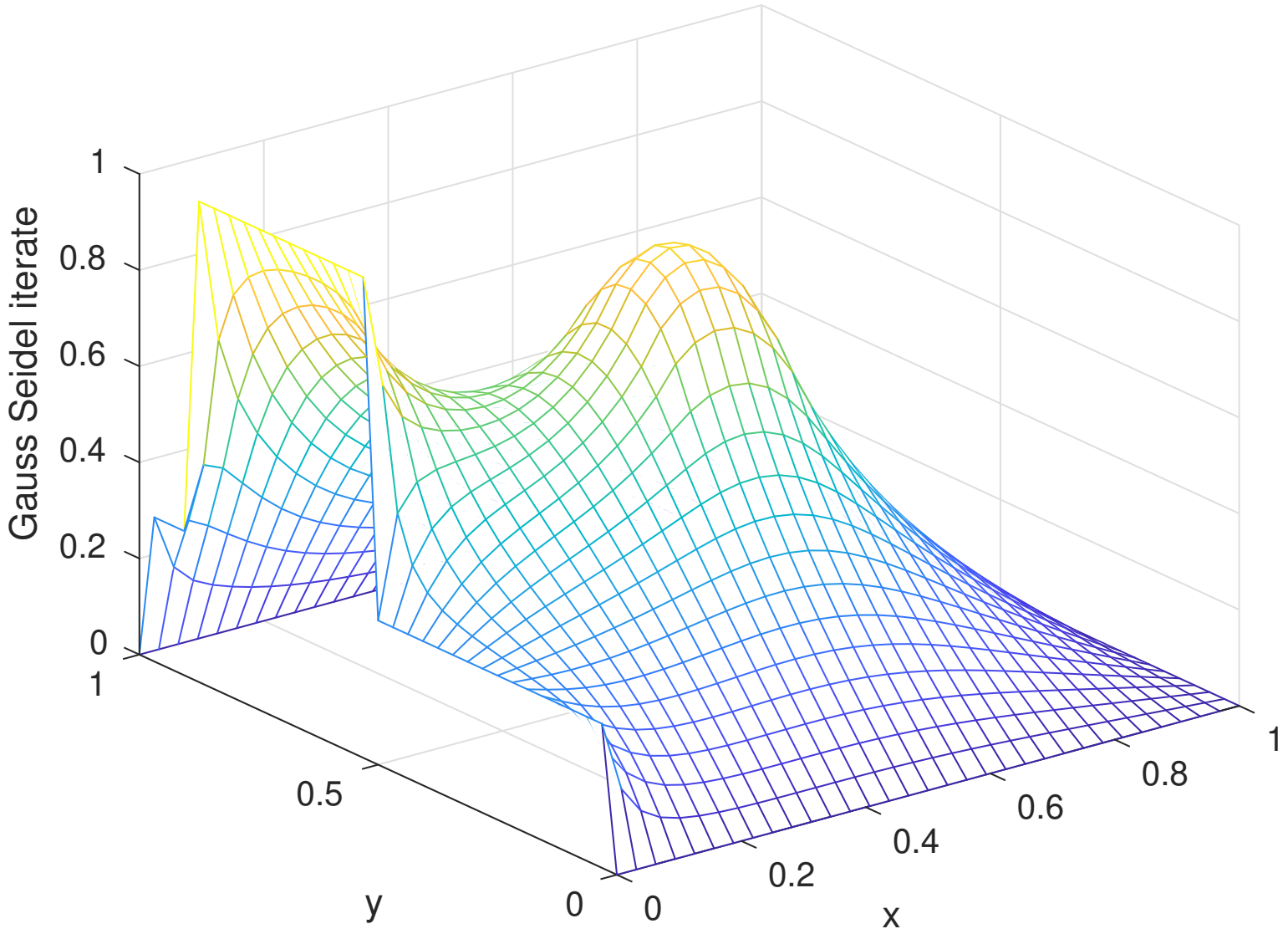
Gauss-Seidel iteration 700



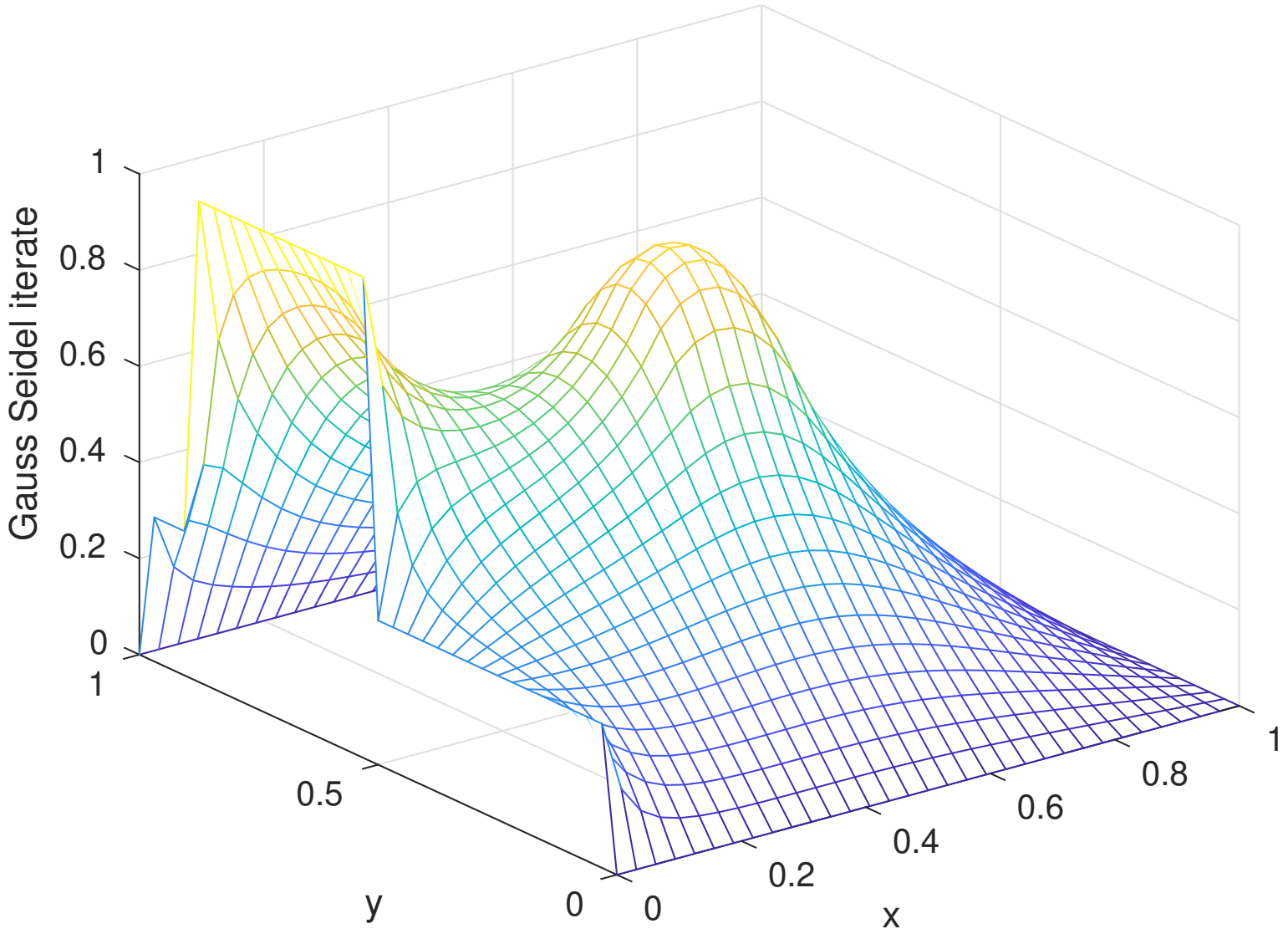
Gauss-Seidel iteration 800



Gauss-Seidel iteration 900



Gauss-Seidel iteration 1000



Stationary Iterations in Modern Notation

For a linear system

$$A\boldsymbol{u} = \boldsymbol{f},$$

one needs a splitting of $A = M - N$ and then iterates

$$M\boldsymbol{u}^{n+1} = N\boldsymbol{u}^{n} + \boldsymbol{f}$$

Examples:

- Jacobi: $M = \mathrm{diag}(A)$
- Gauss-Seidel: $M = \mathrm{tril}(A)$

Can these iterations be made faster ?

Jacobi (1845): "Man kann aber [...] durch Wiederholung einer leichten Rechnung die Gleichungen in andere umformen, in welchen der erwähnte Uebelstand immer weniger hervortritt".

David Young's (1950) PhD-thesis: Successive Overrelaxation (SOR)

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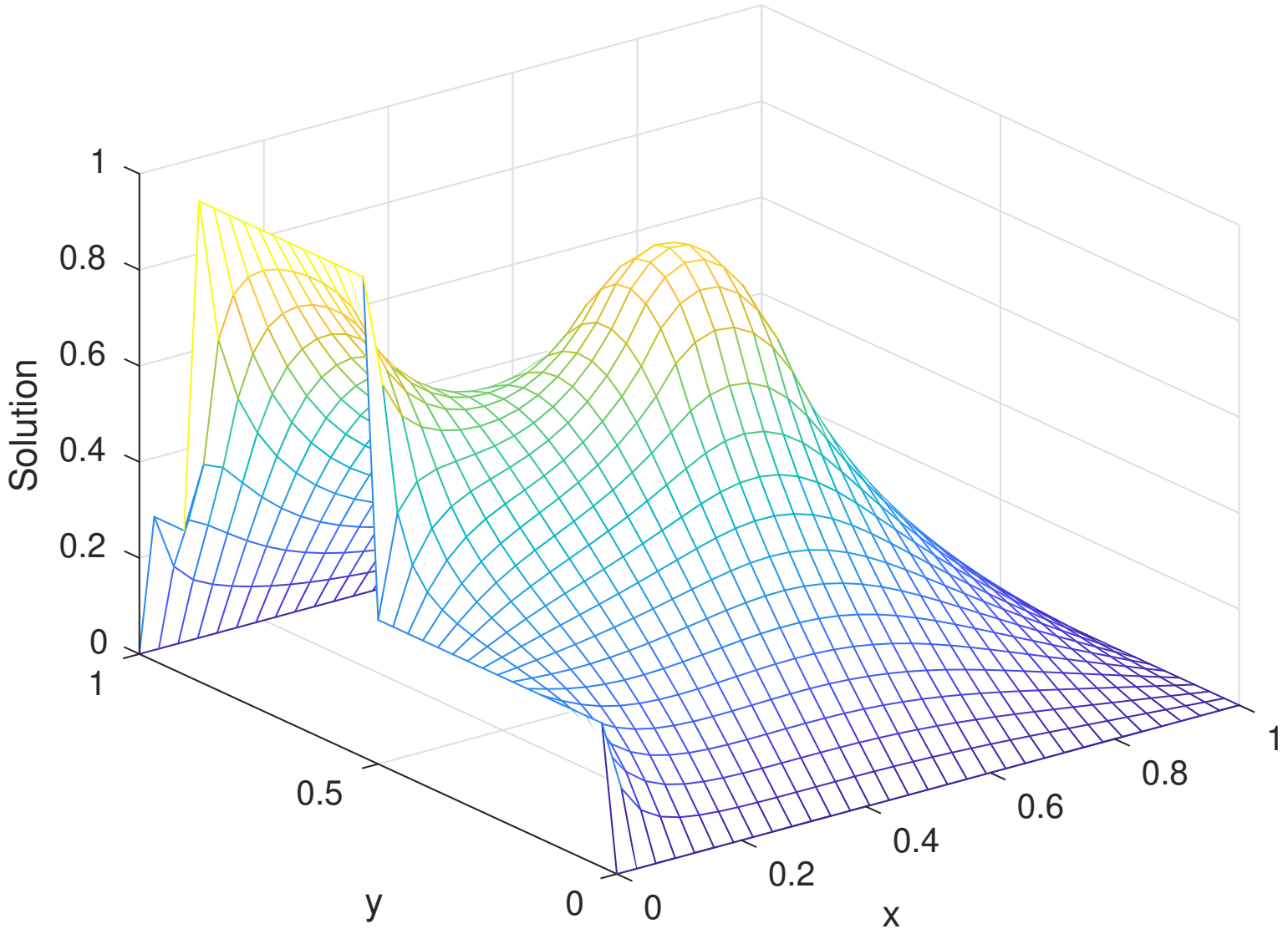
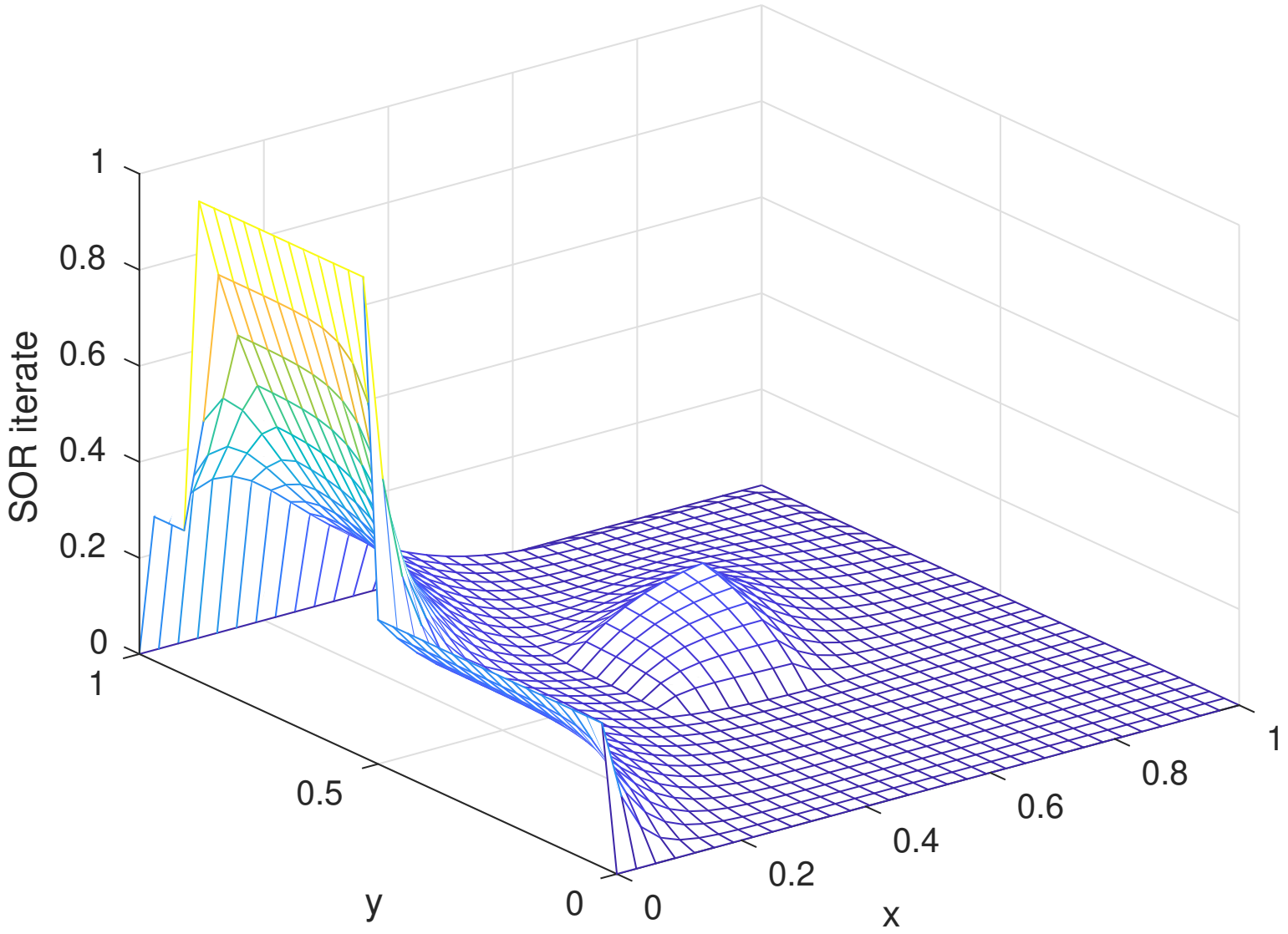
Example SOR: temperature in a room



SOR iteration 1



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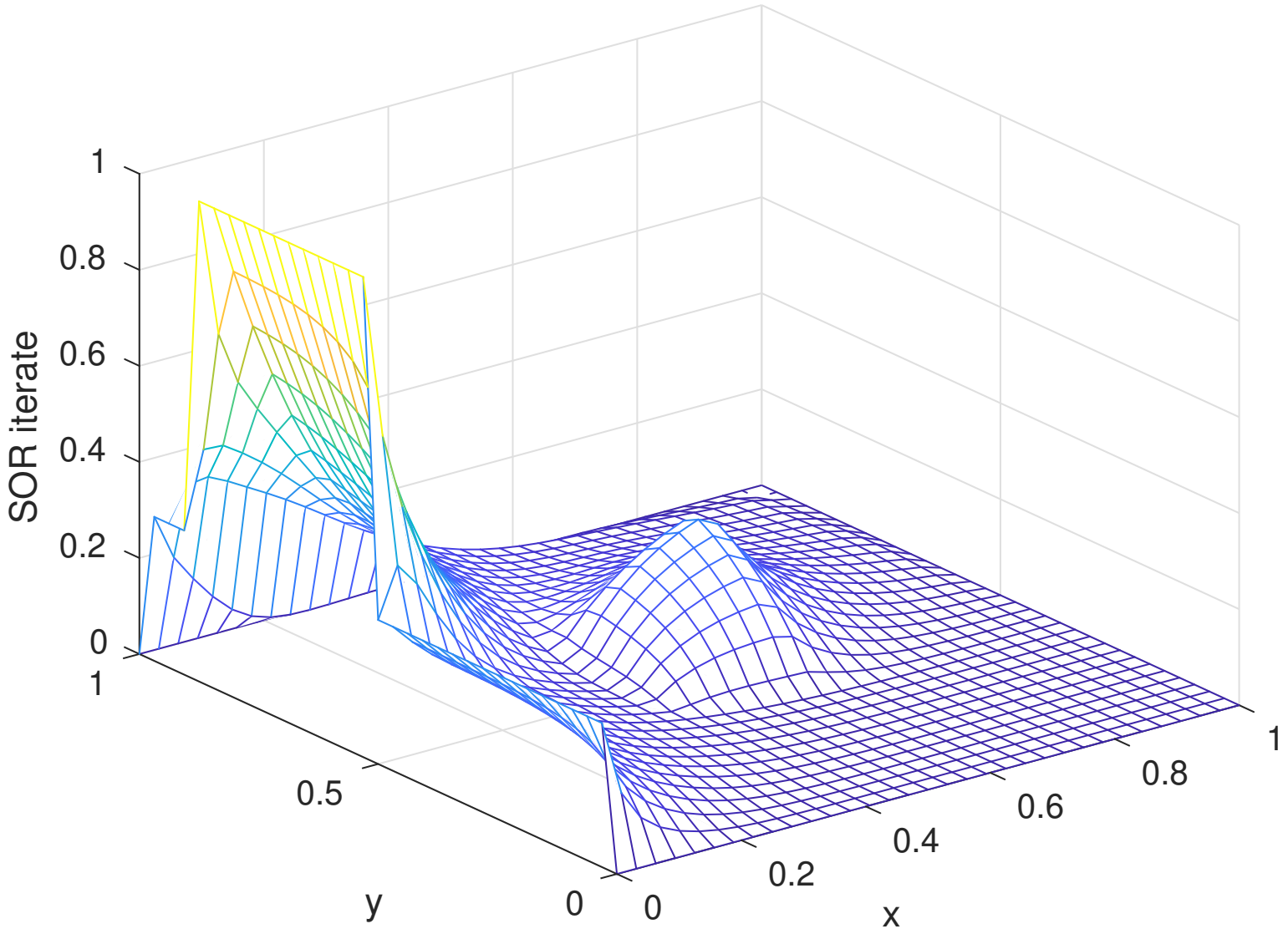
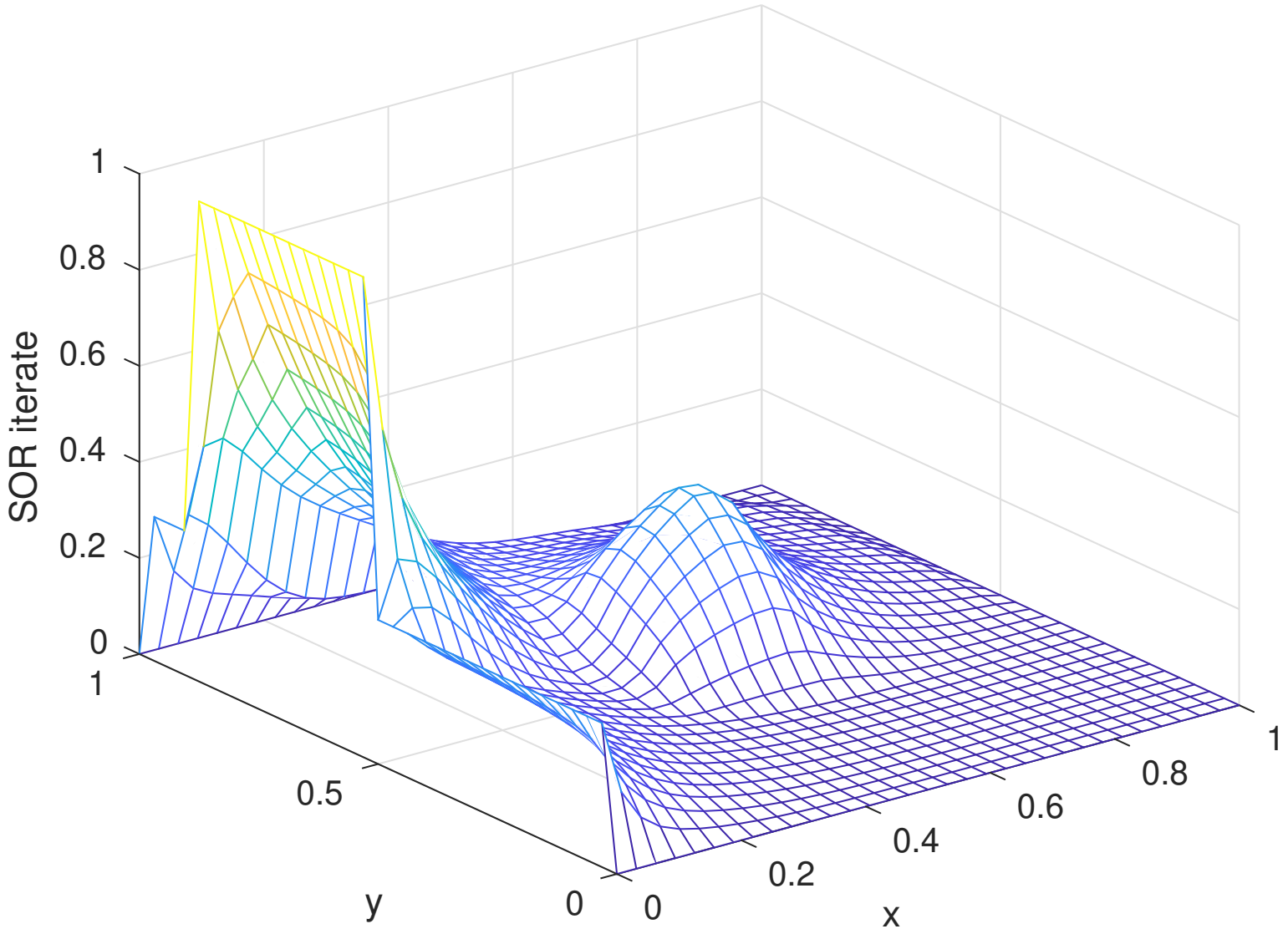
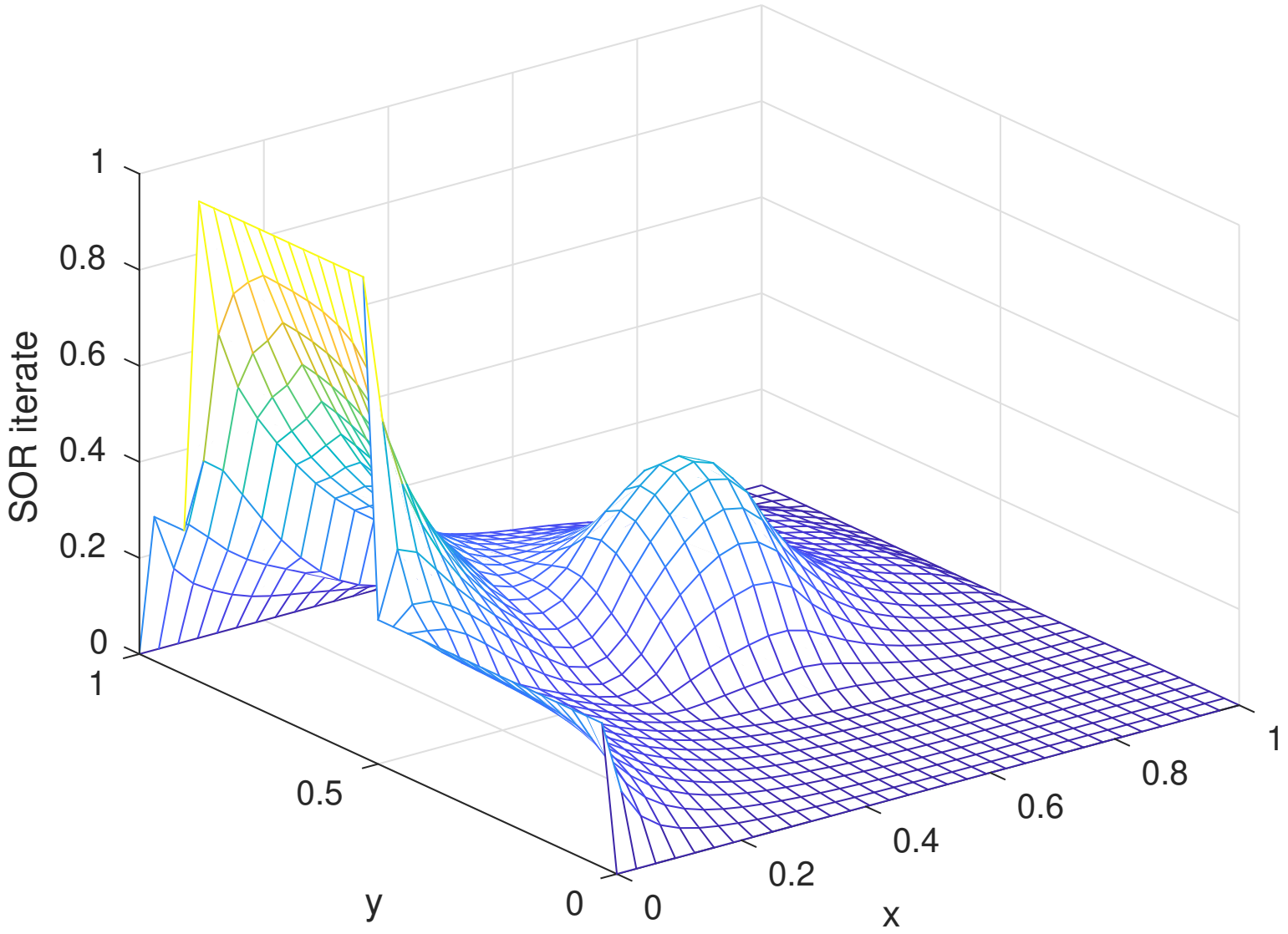
SOR iteration 2



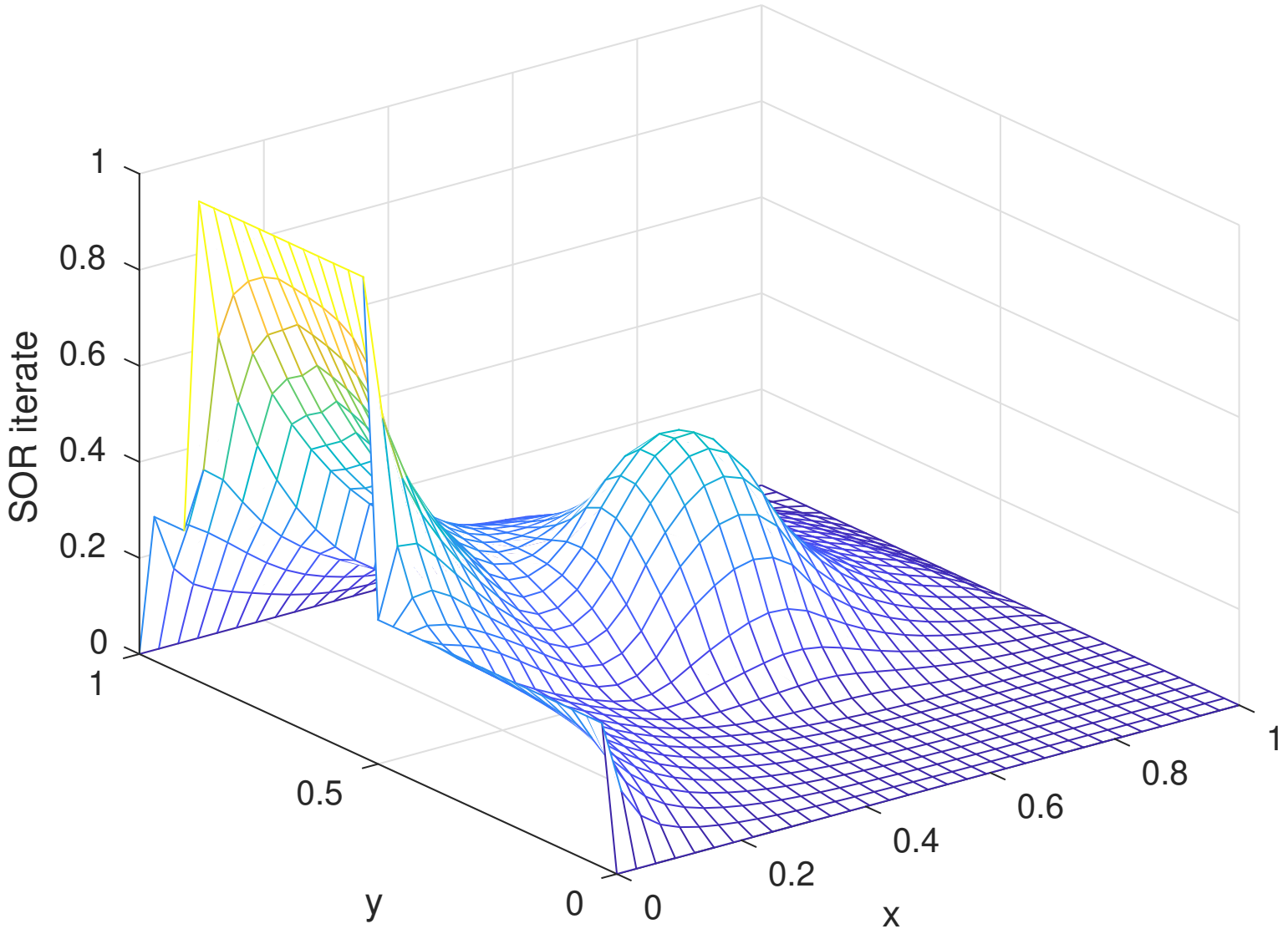
SOR iteration 3



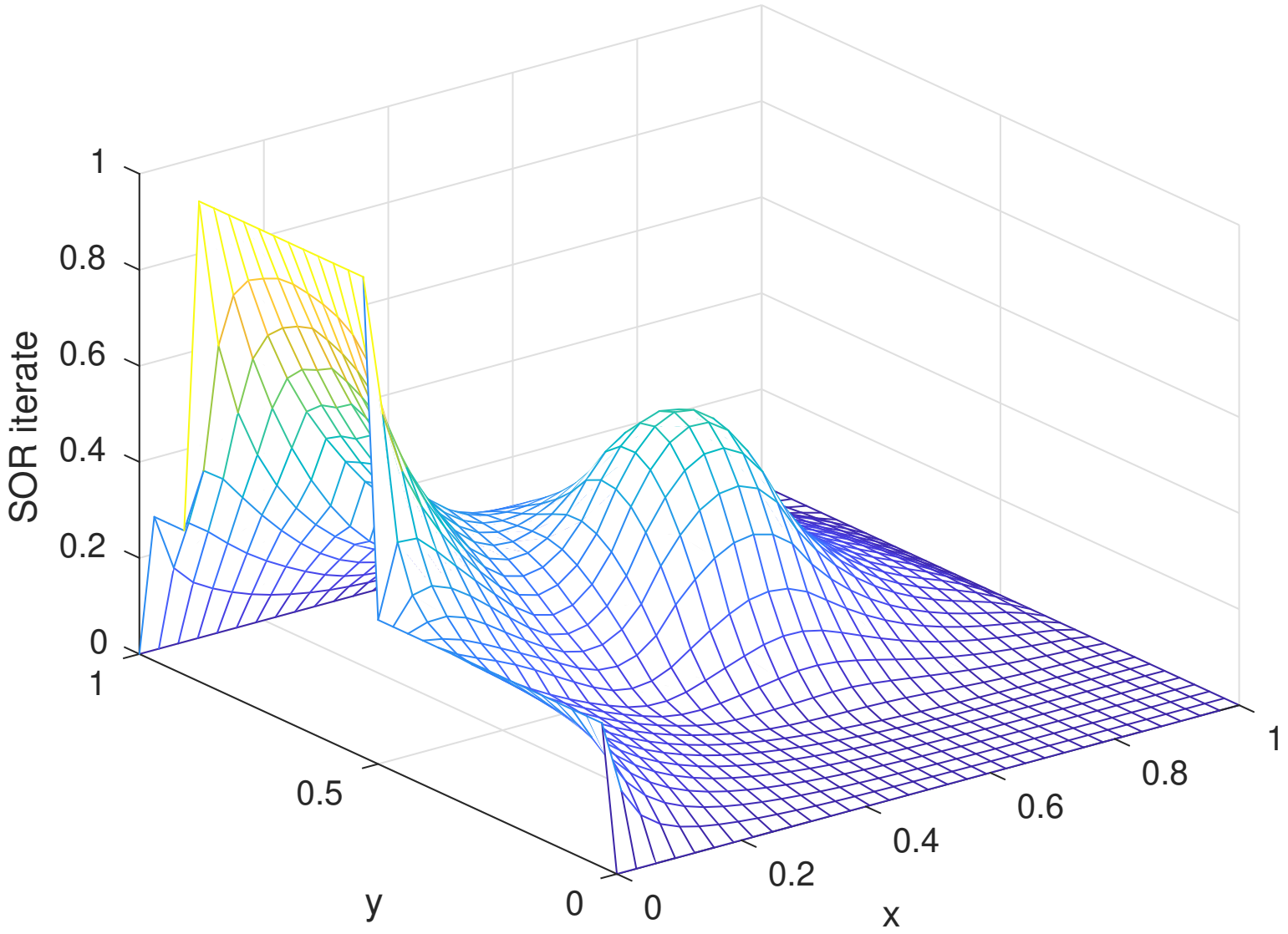
SOR iteration 4



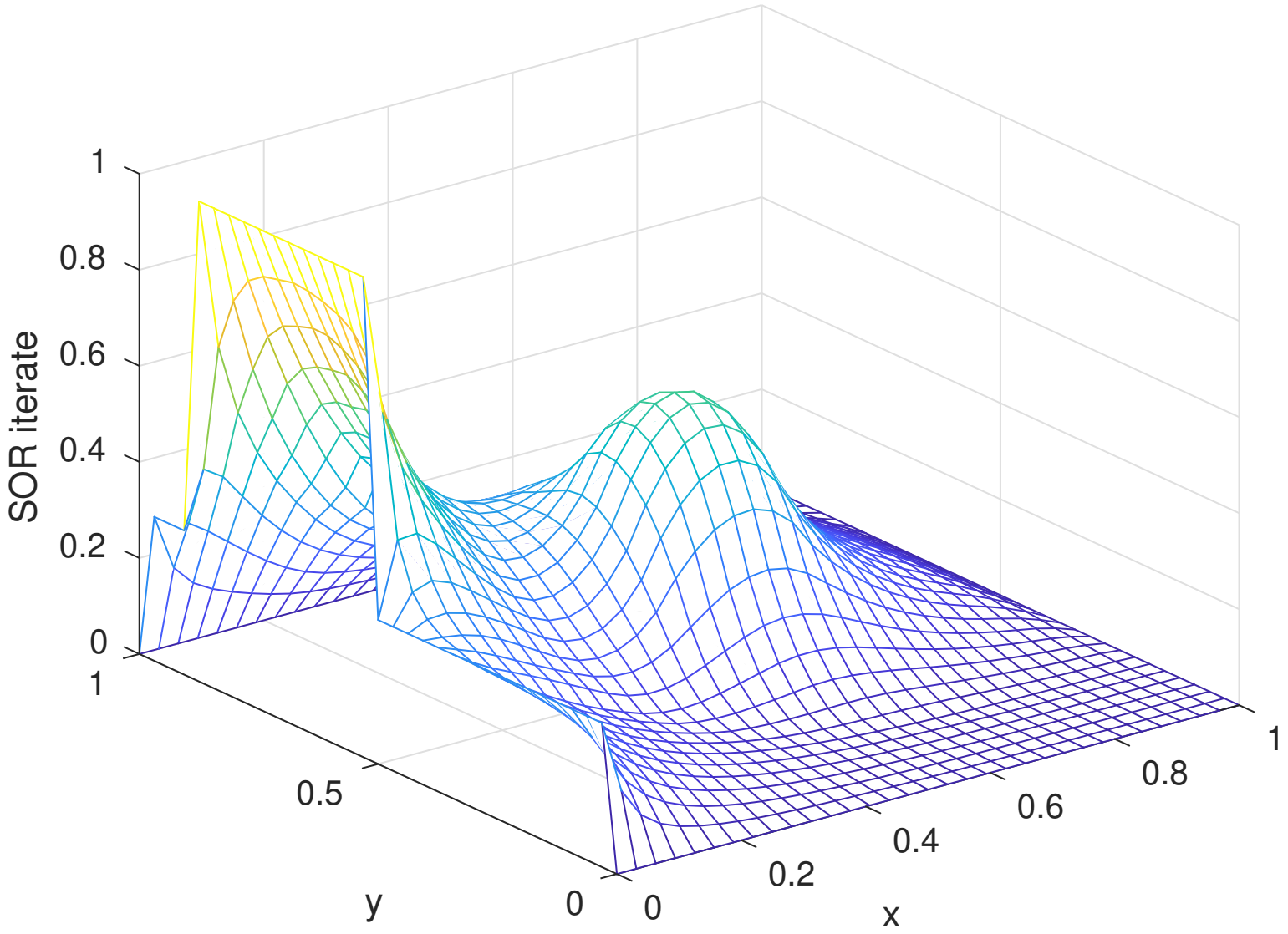
SOR iteration 5



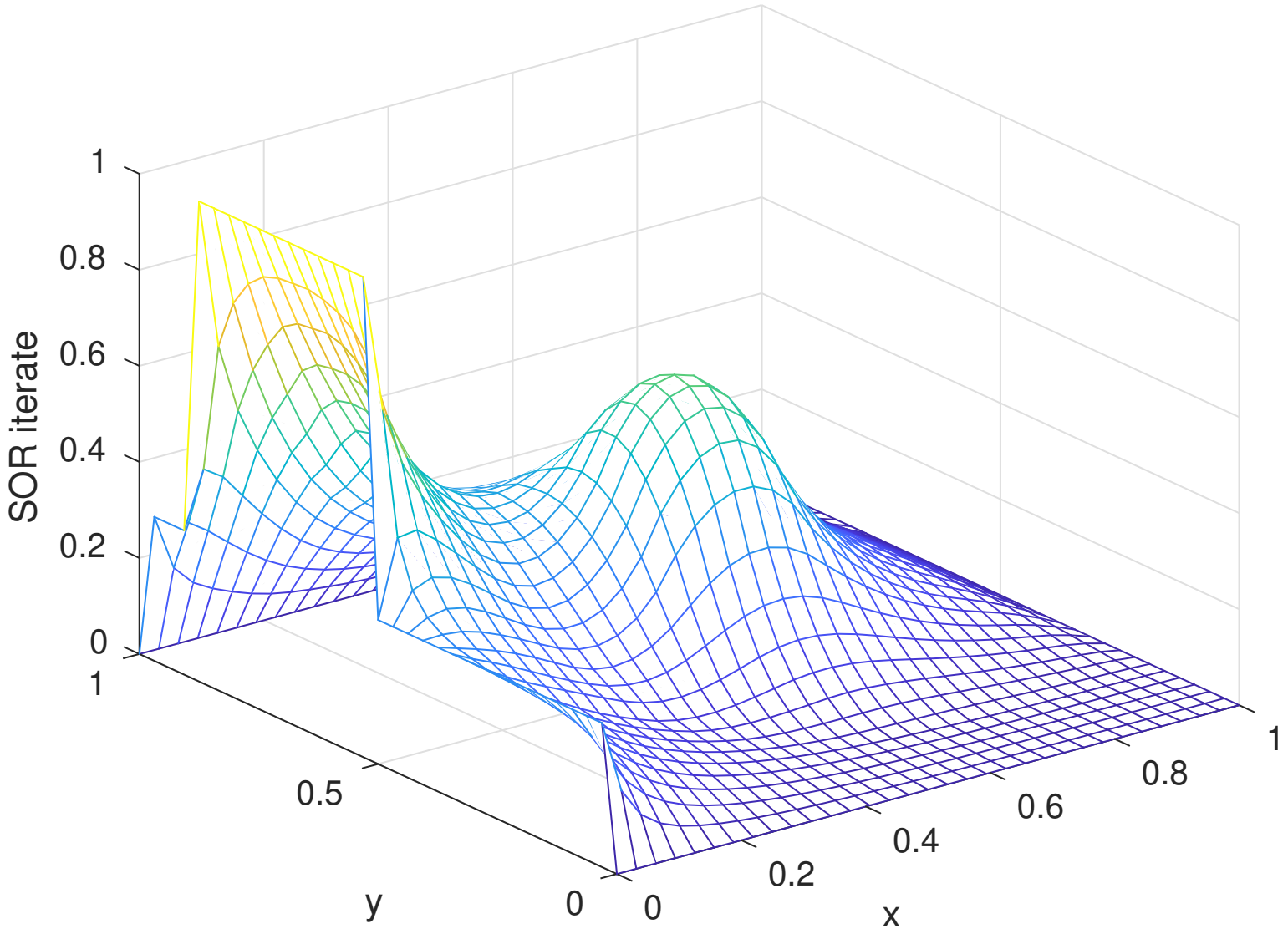
SOR iteration 6



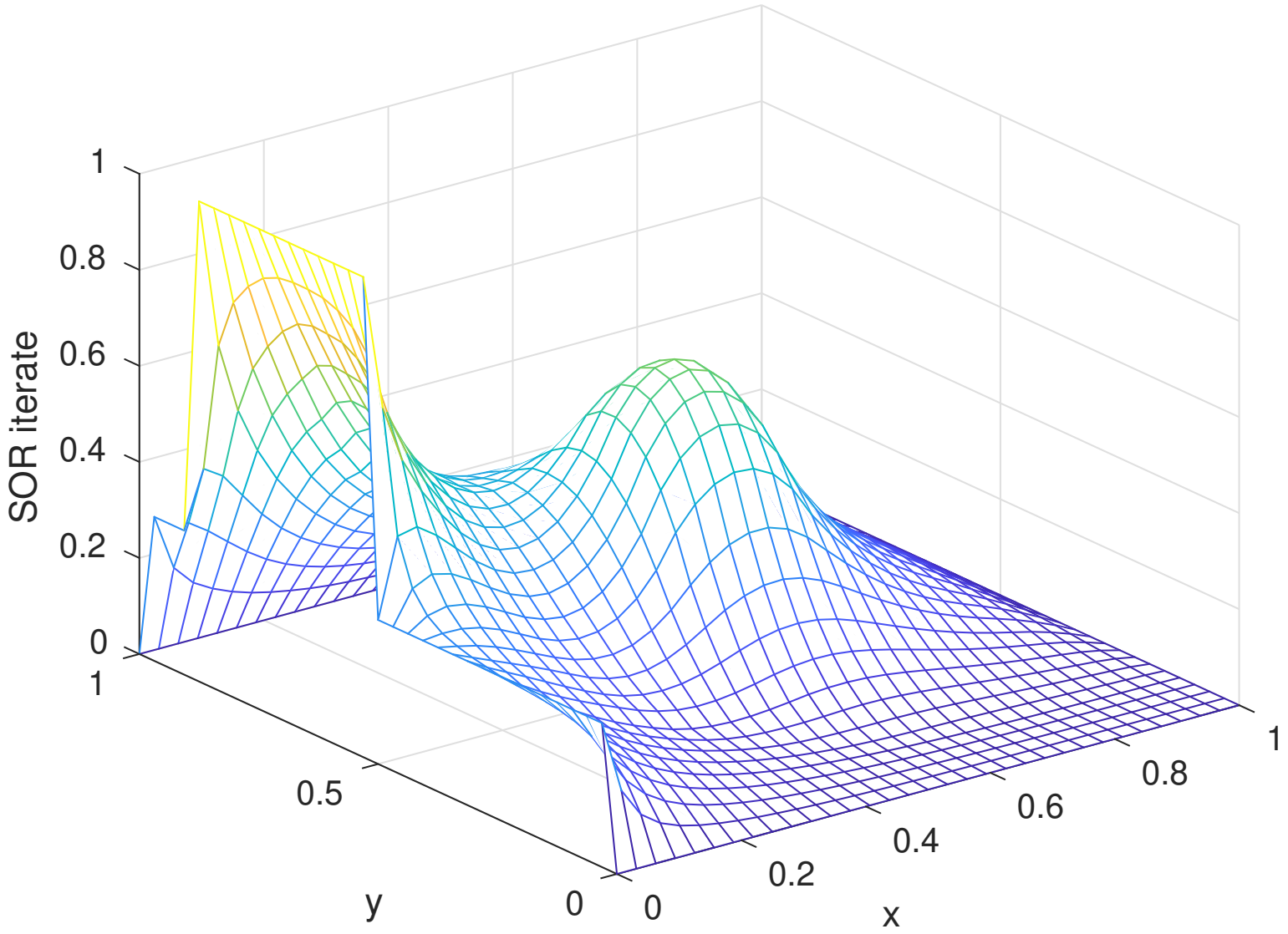
SOR iteration 7



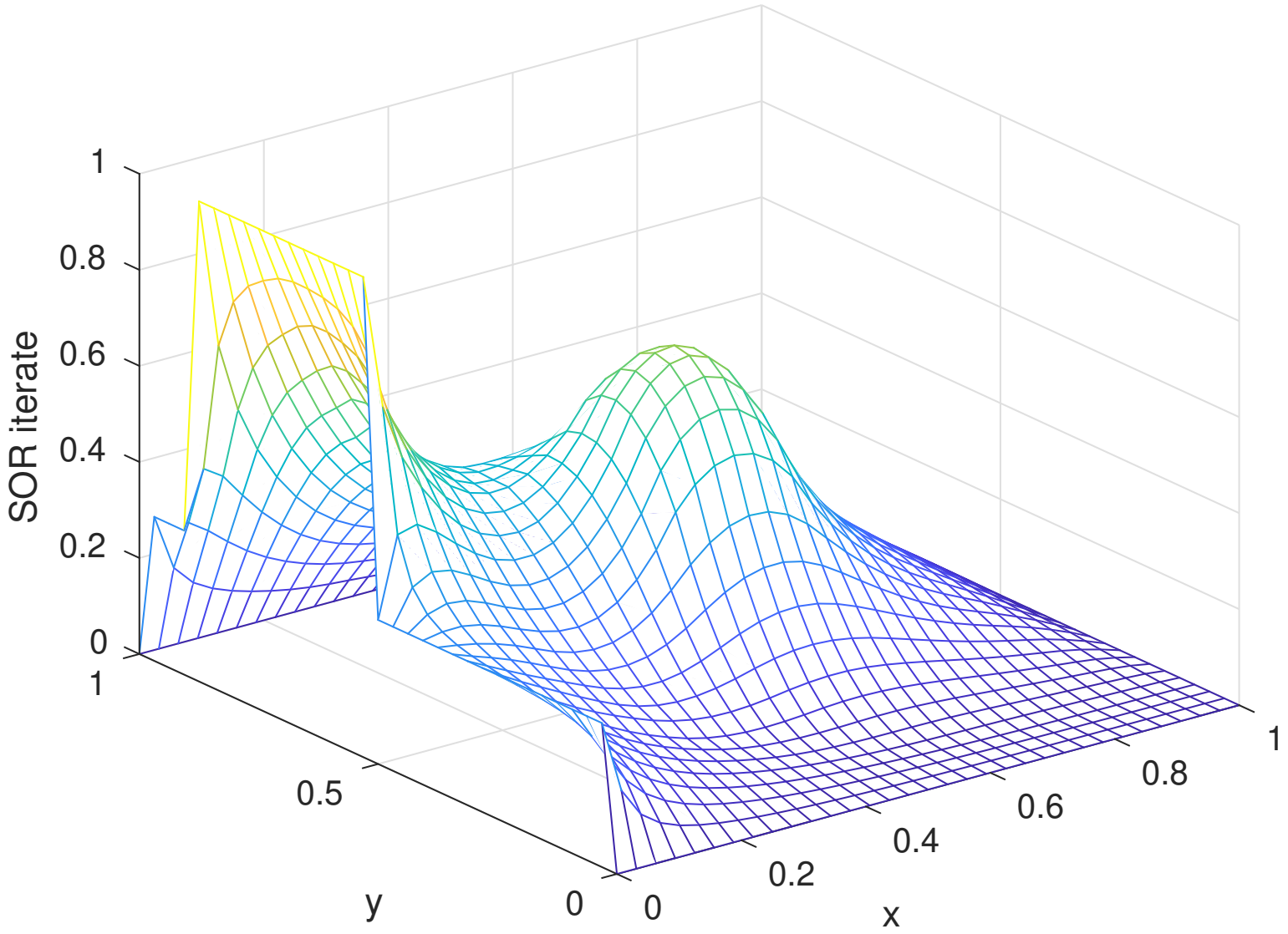
SOR iteration 8



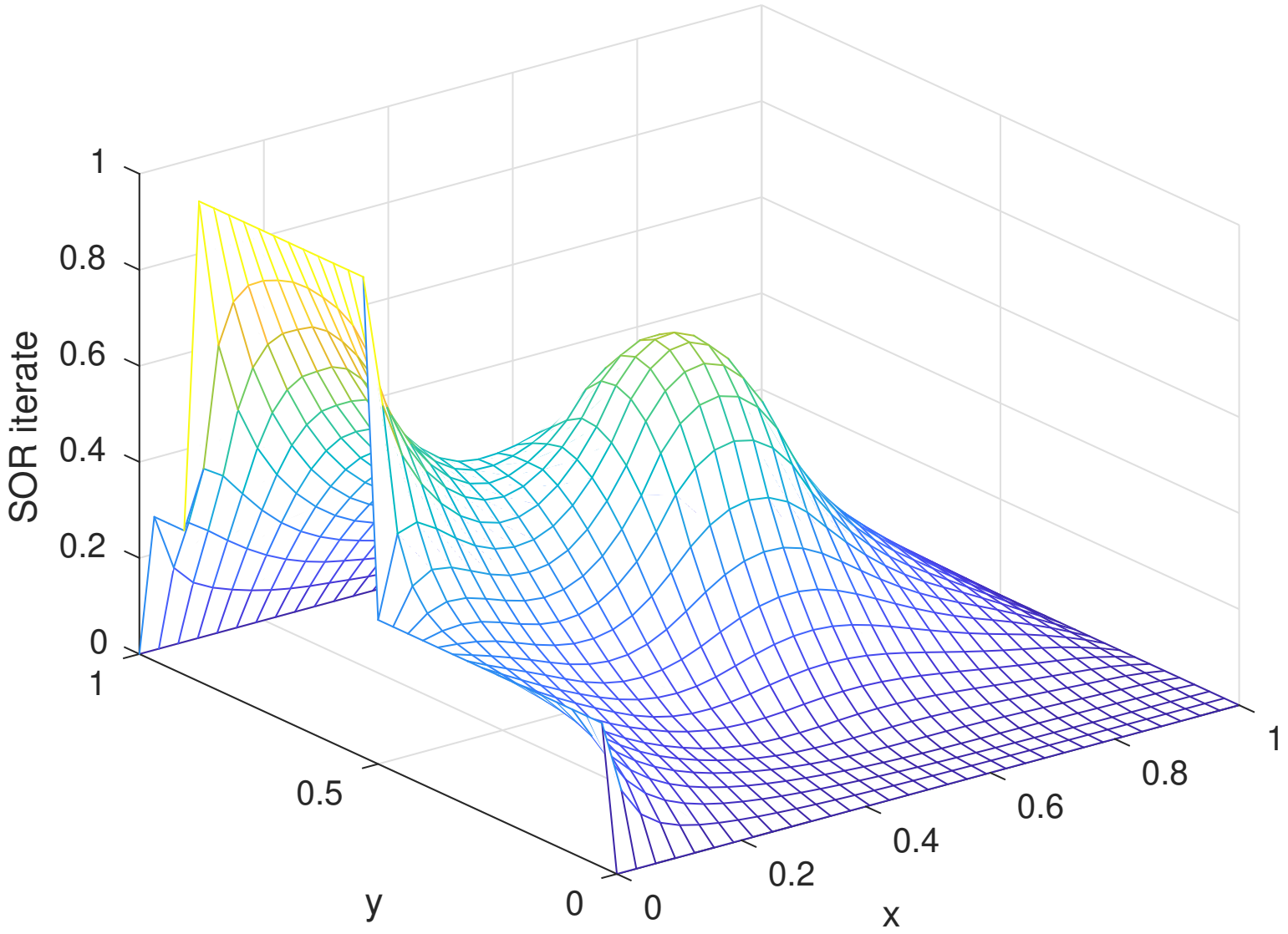
SOR iteration 9



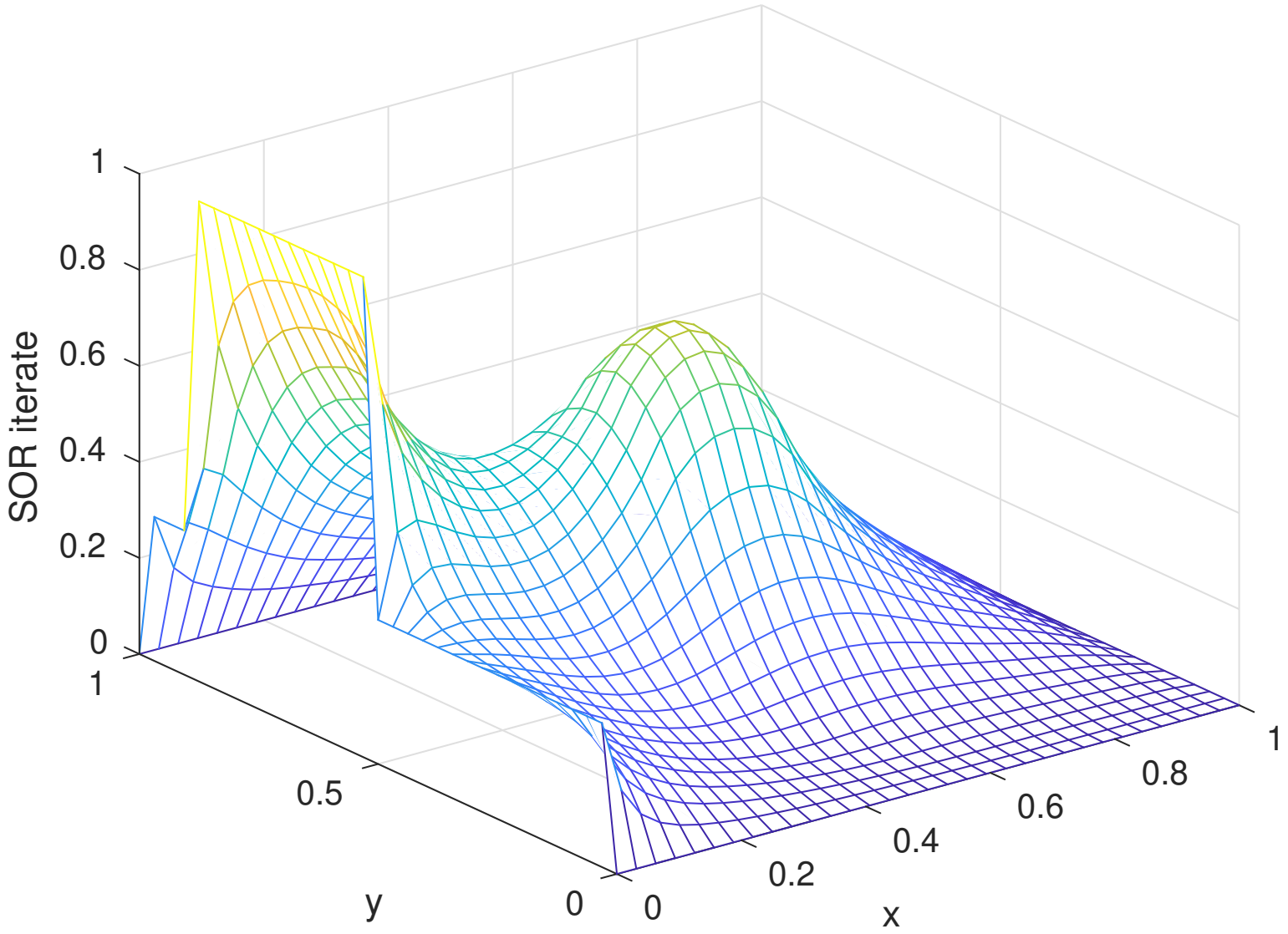
SOR iteration 10



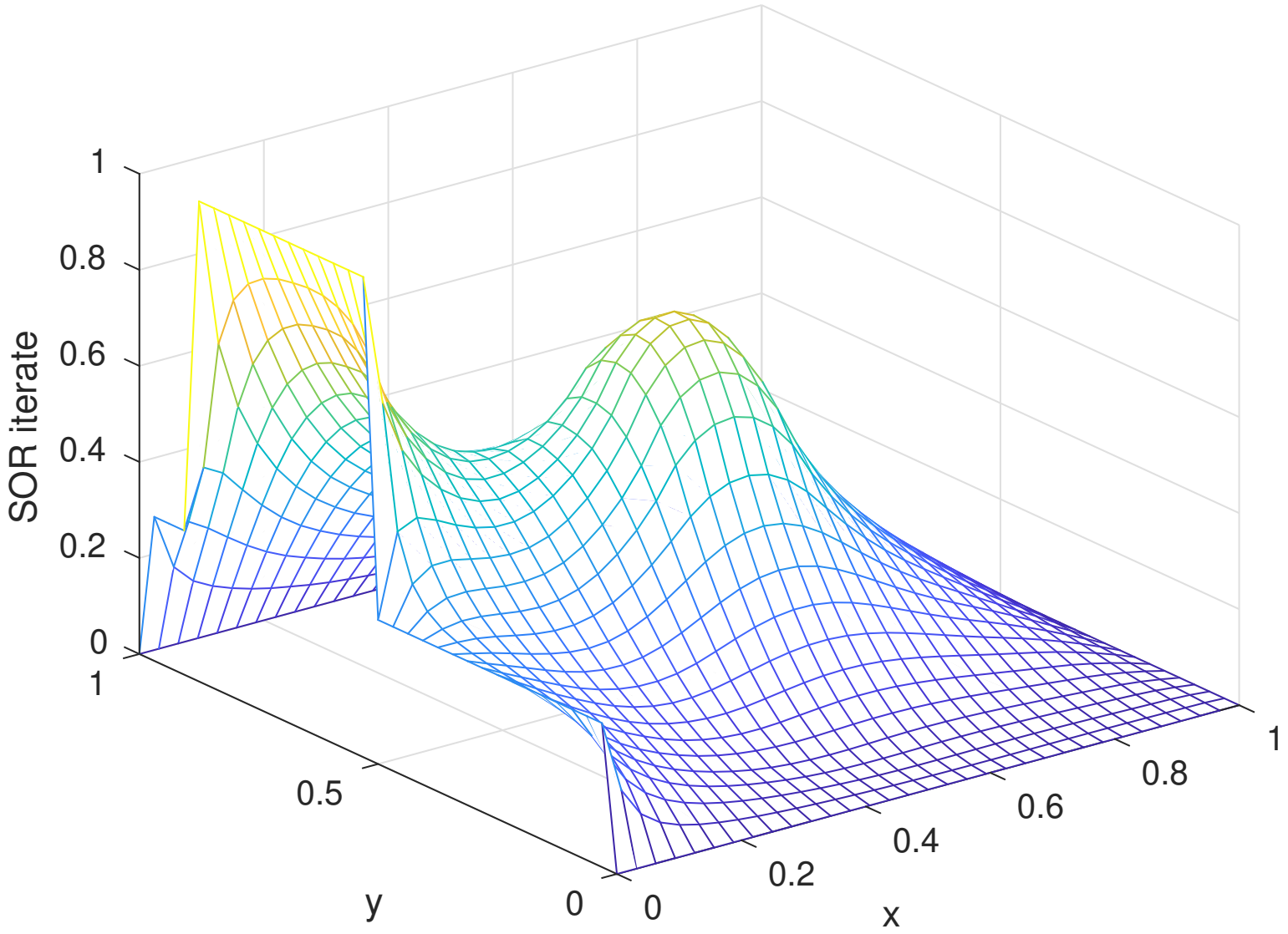
SOR iteration 11



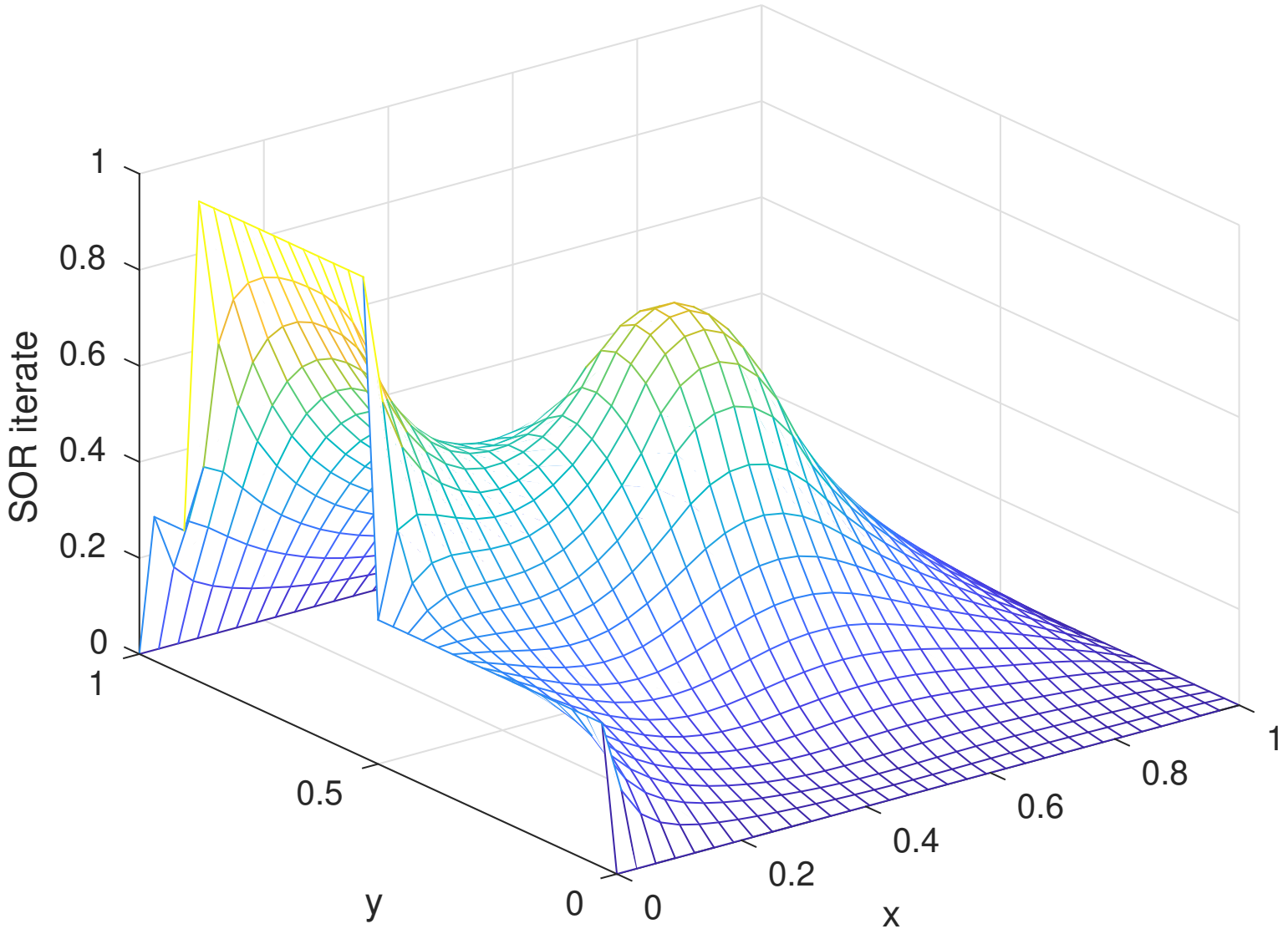
SOR iteration 12



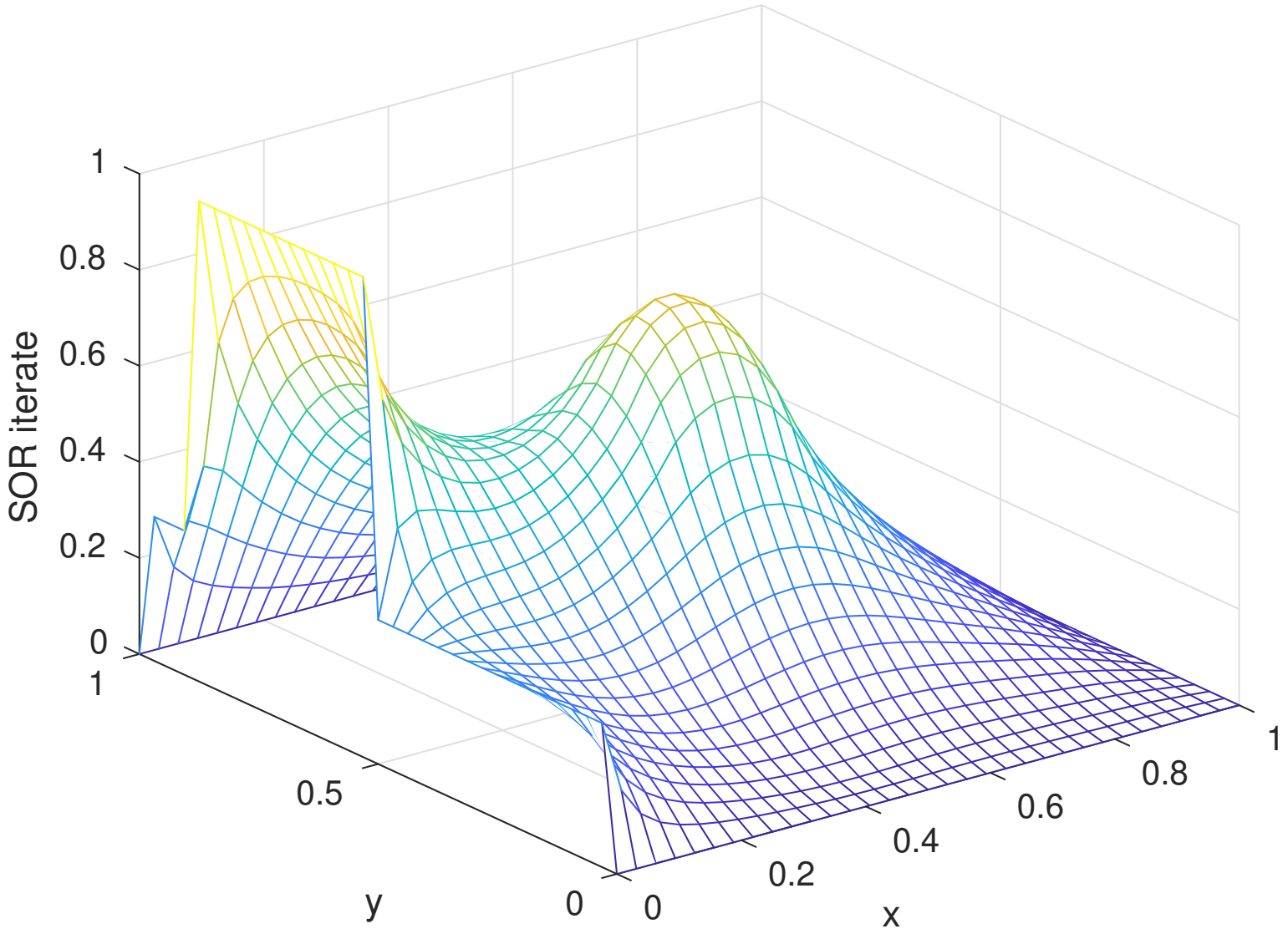
SOR iteration 13



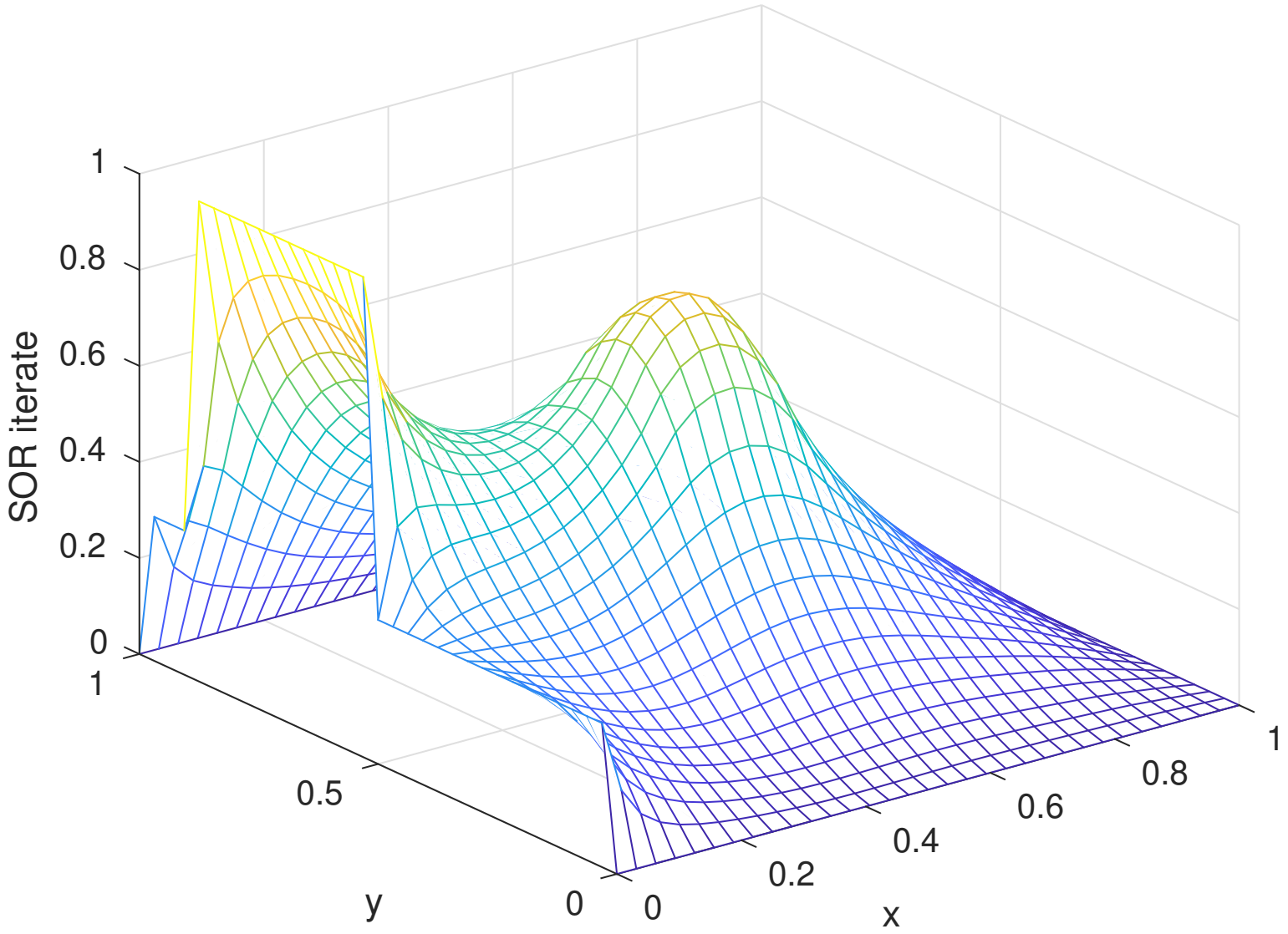
SOR iteration 14



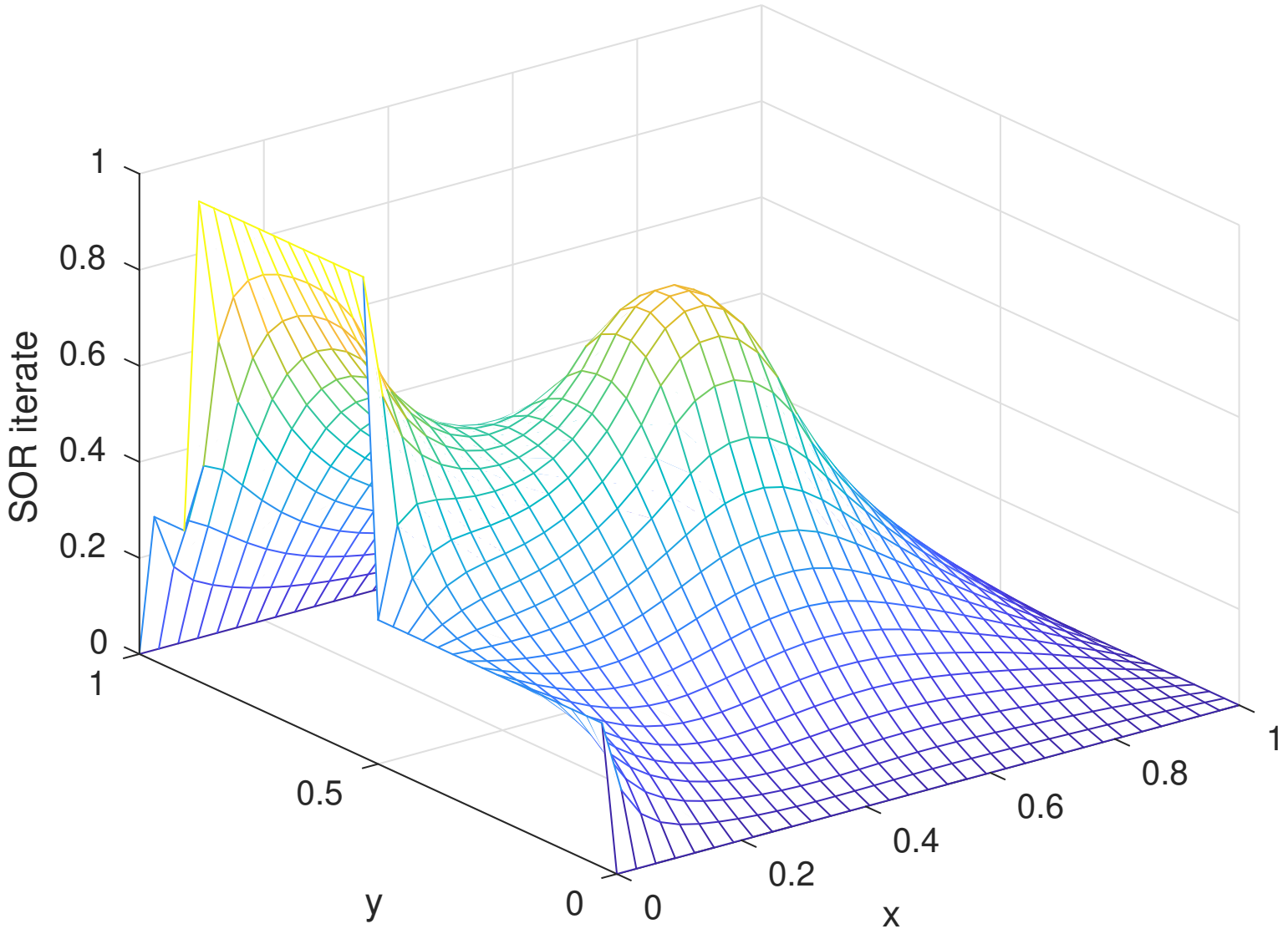
SOR iteration 15



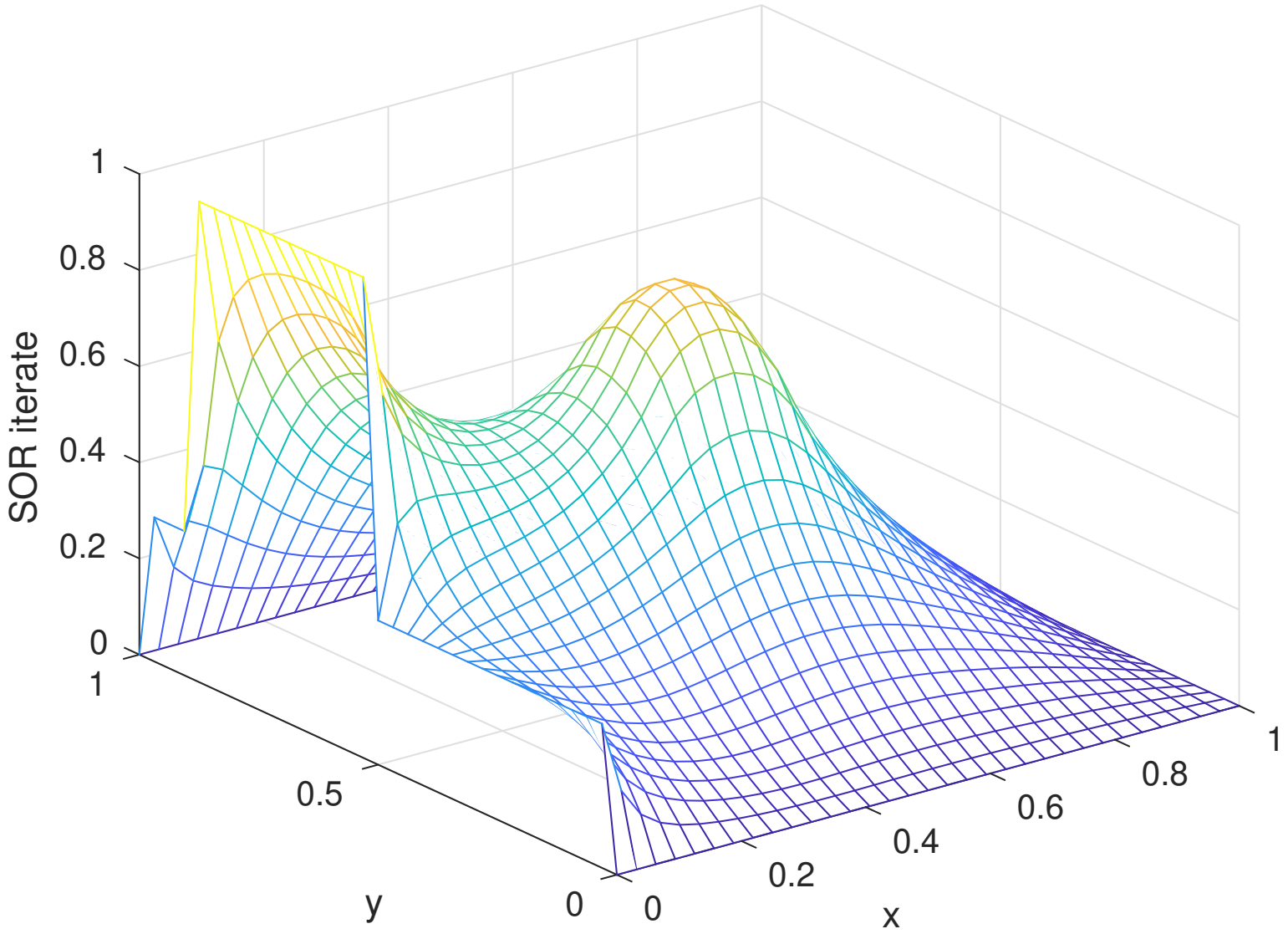
SOR iteration 16



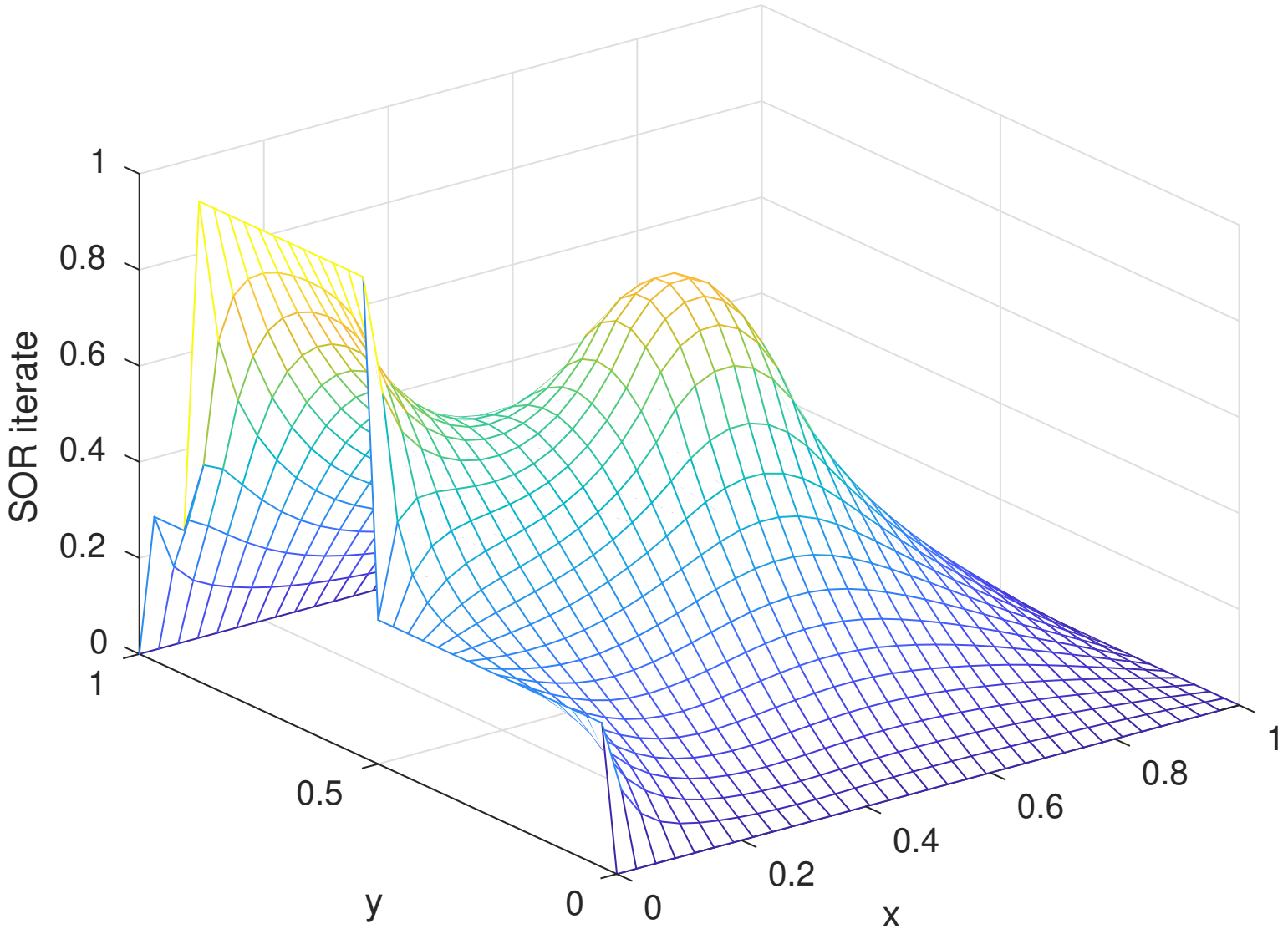
SOR iteration 17



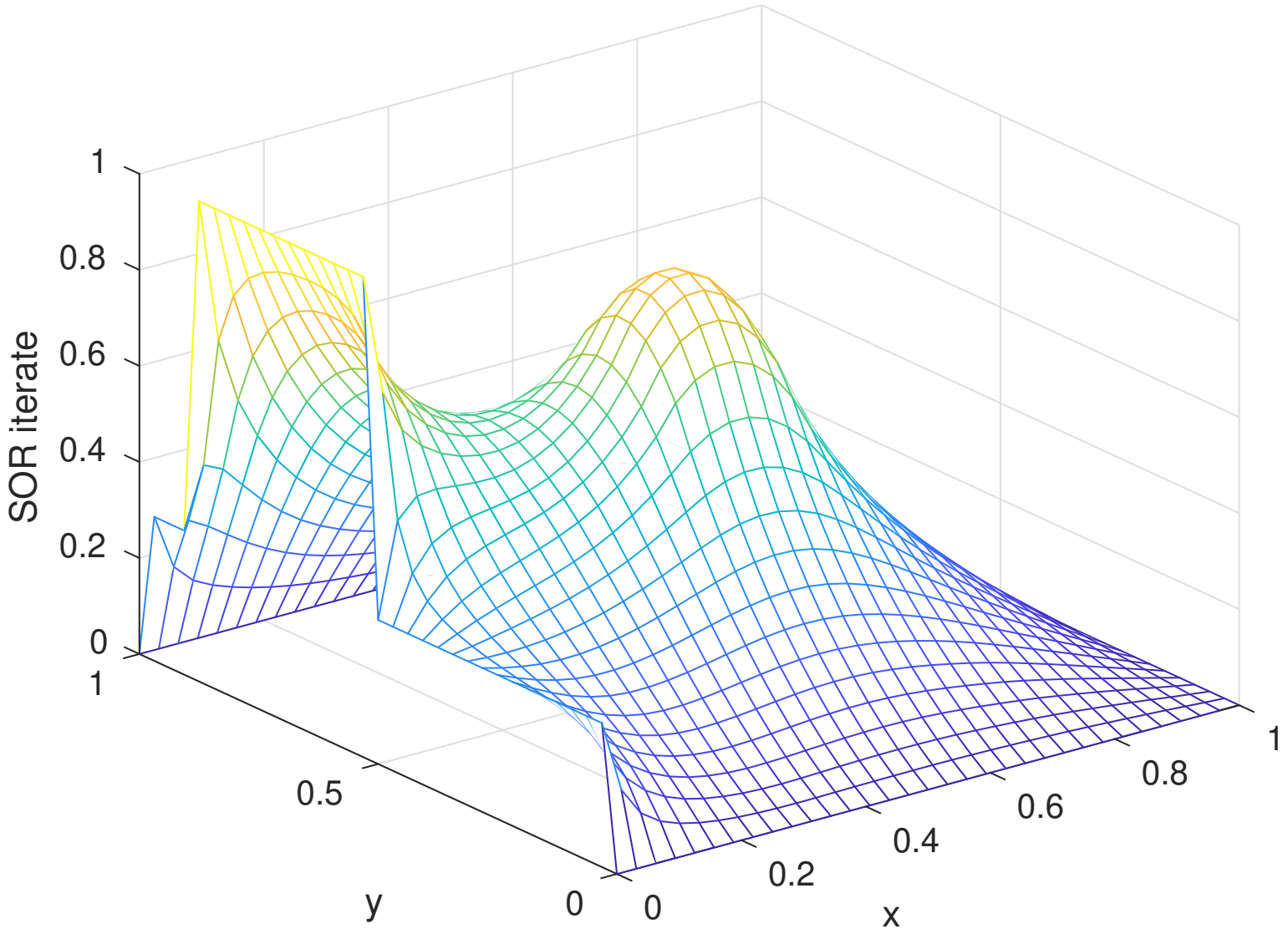
SOR iteration 18



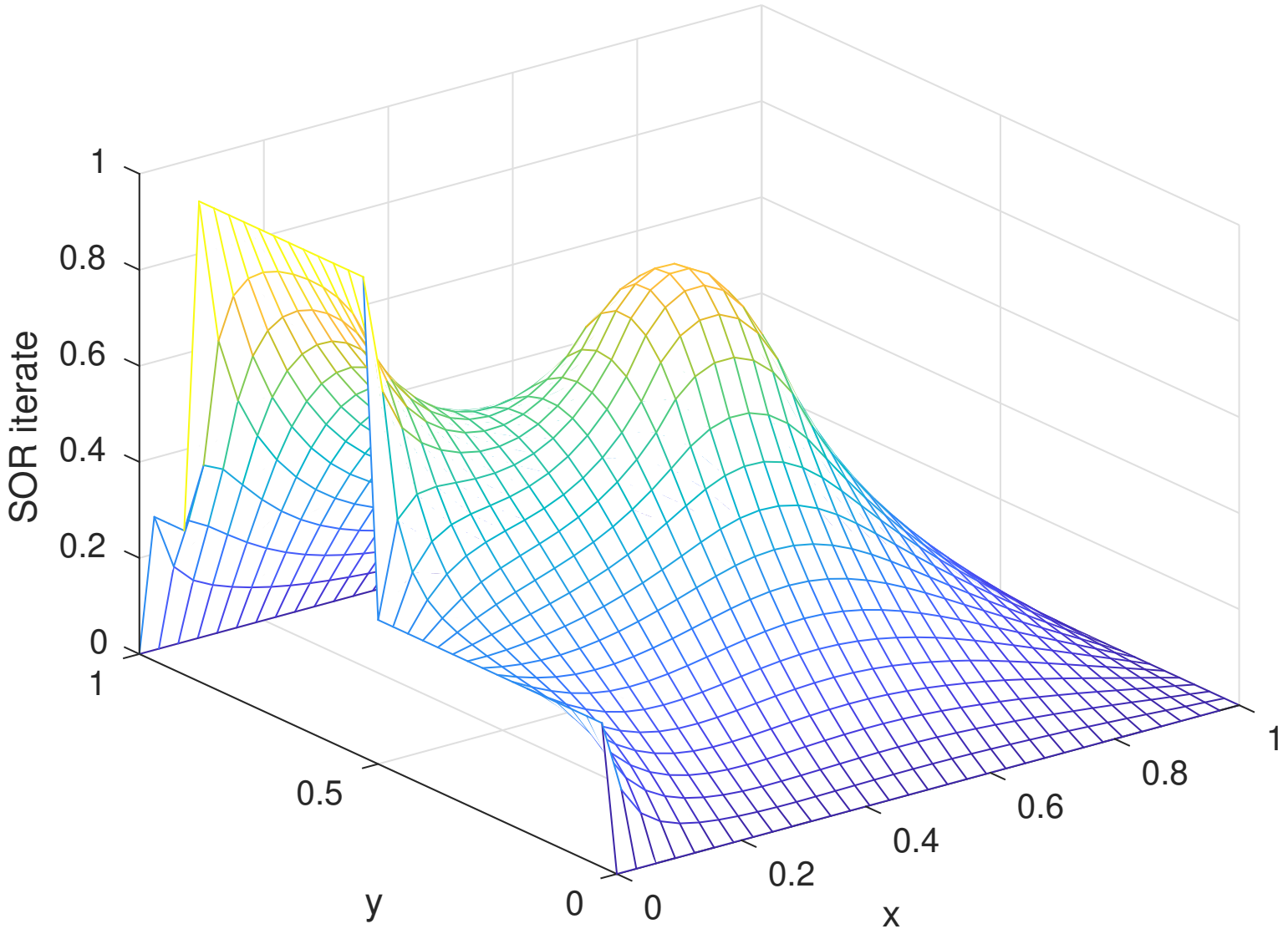
SOR iteration 19



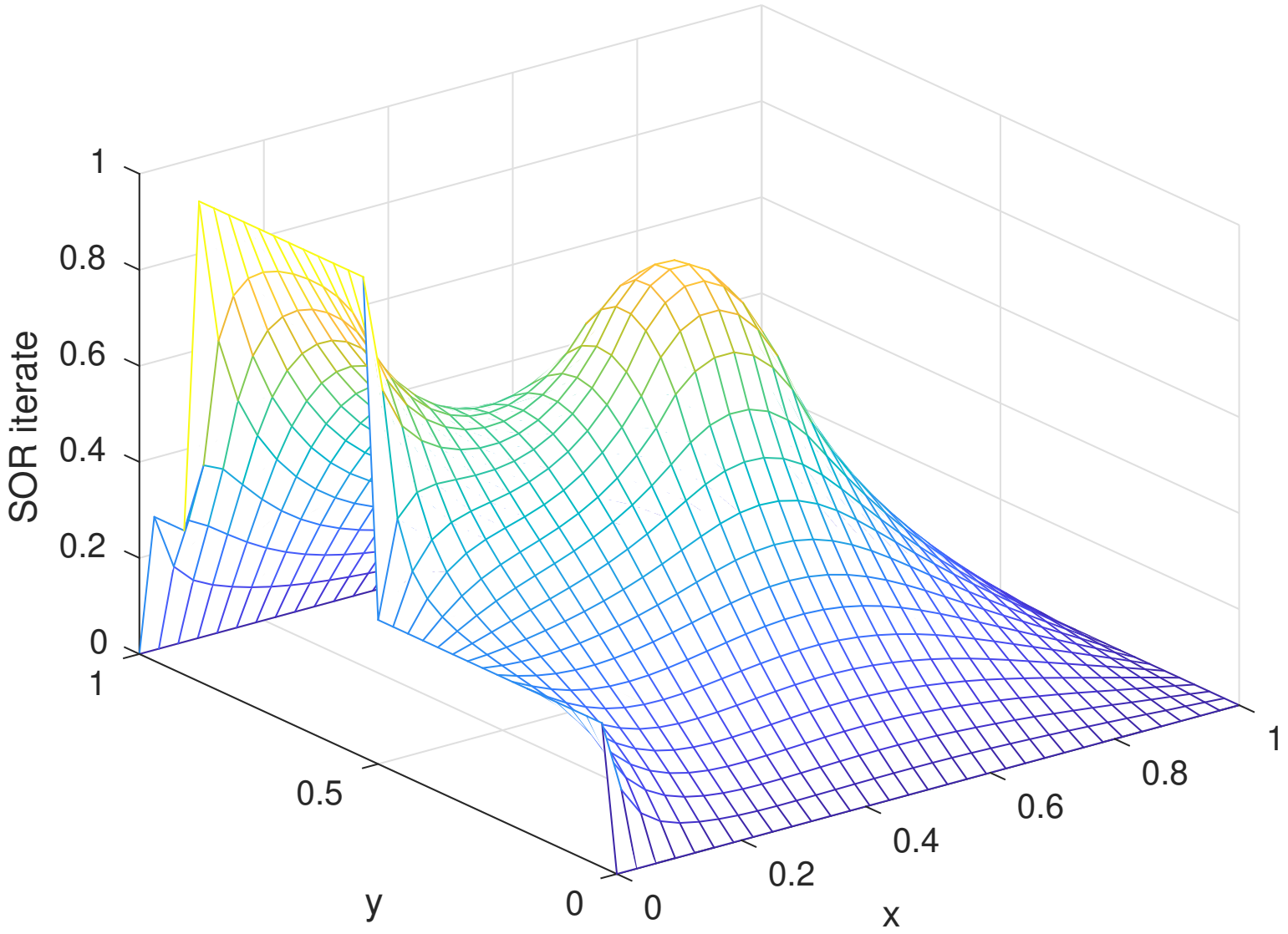
SOR iteration 20



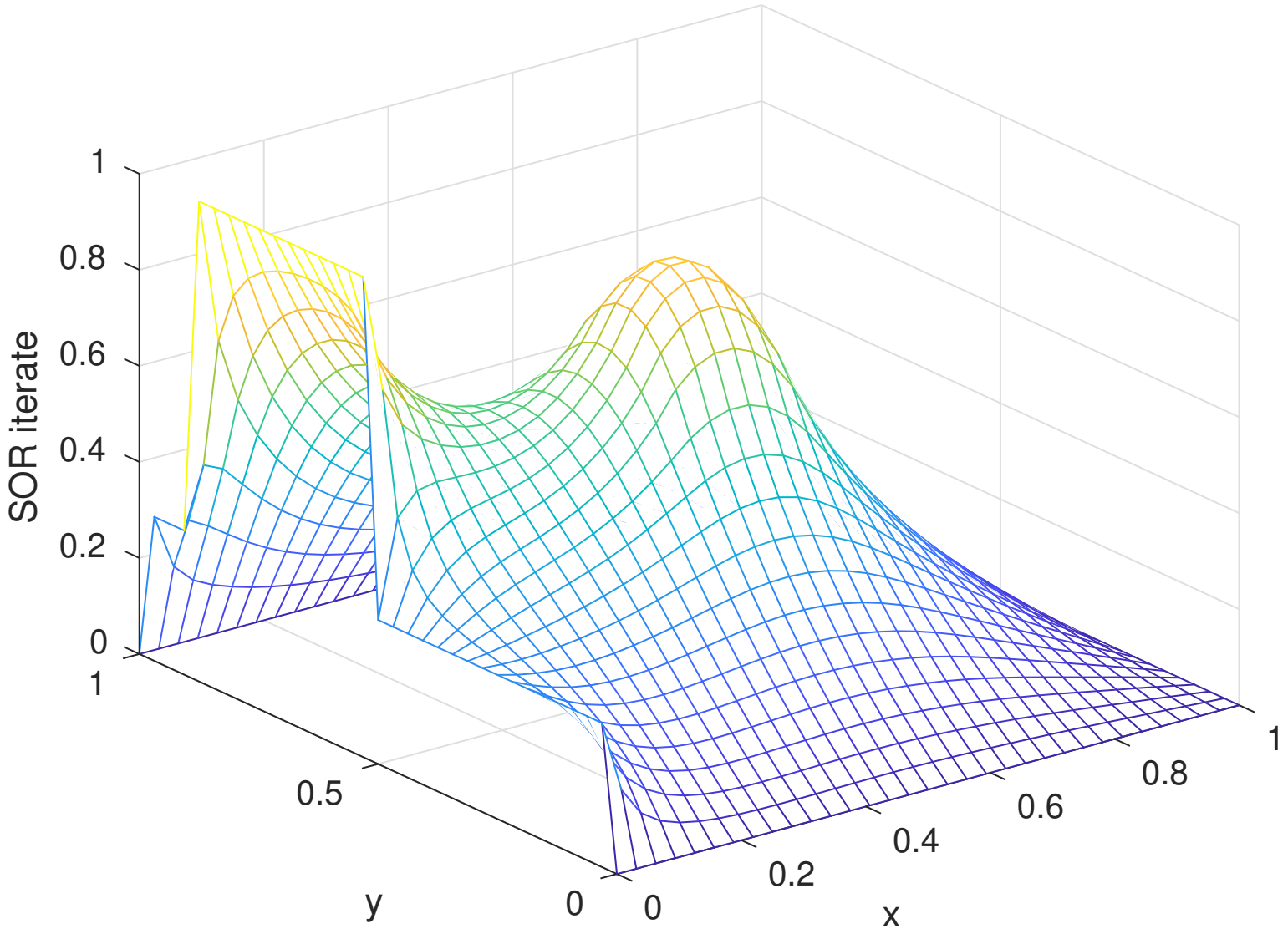
SOR iteration 21



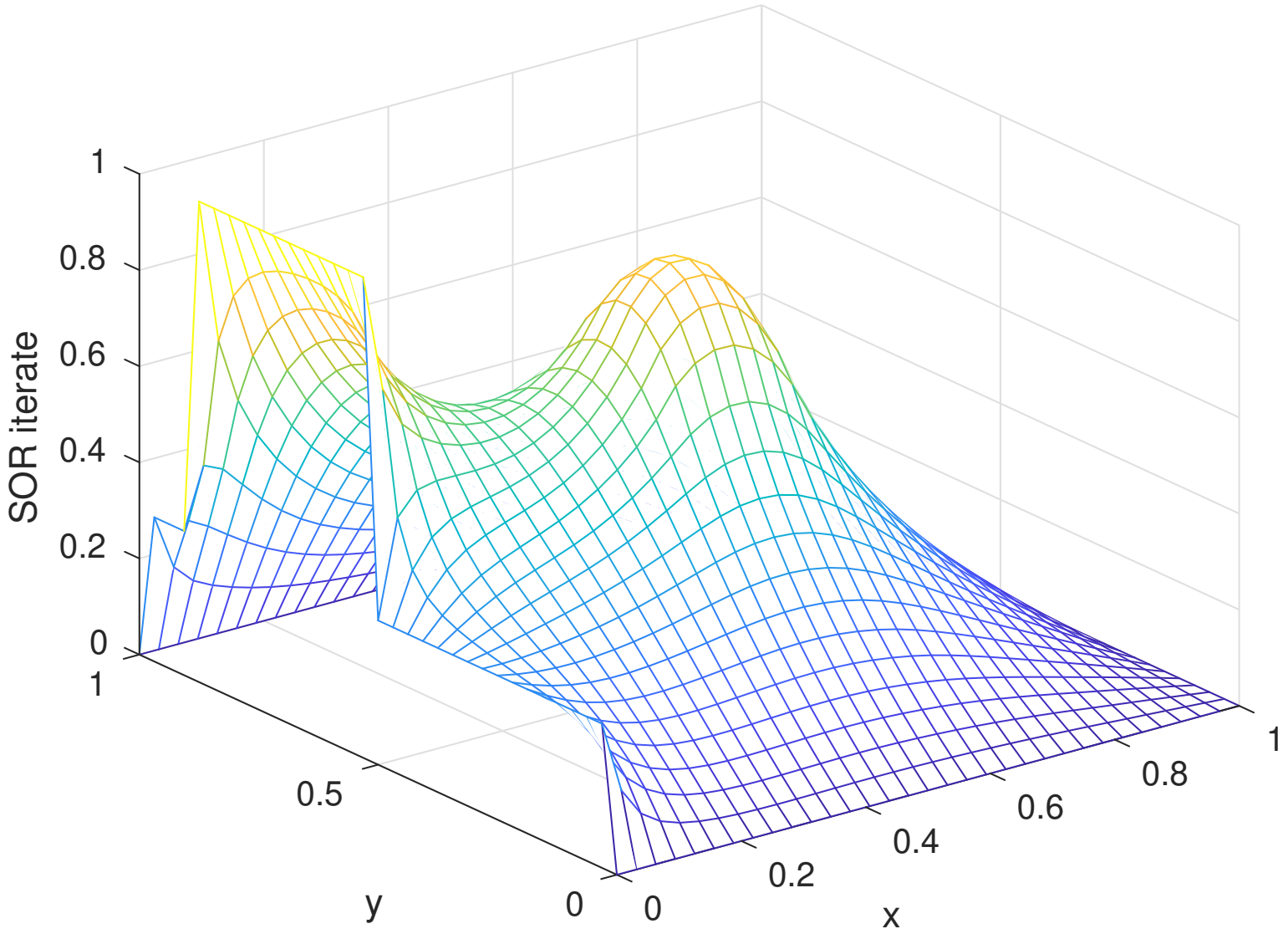
SOR iteration 22



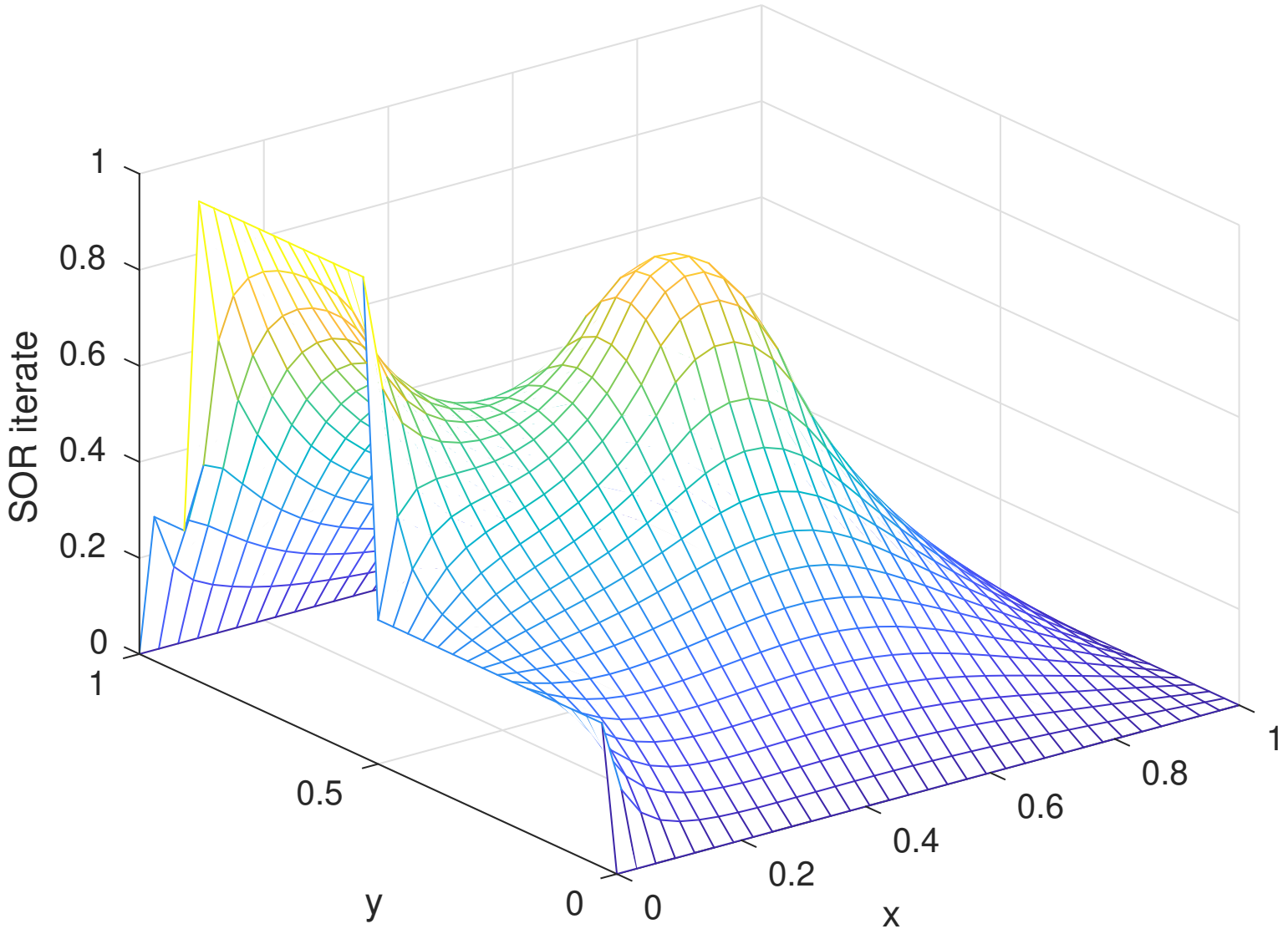
SOR iteration 23



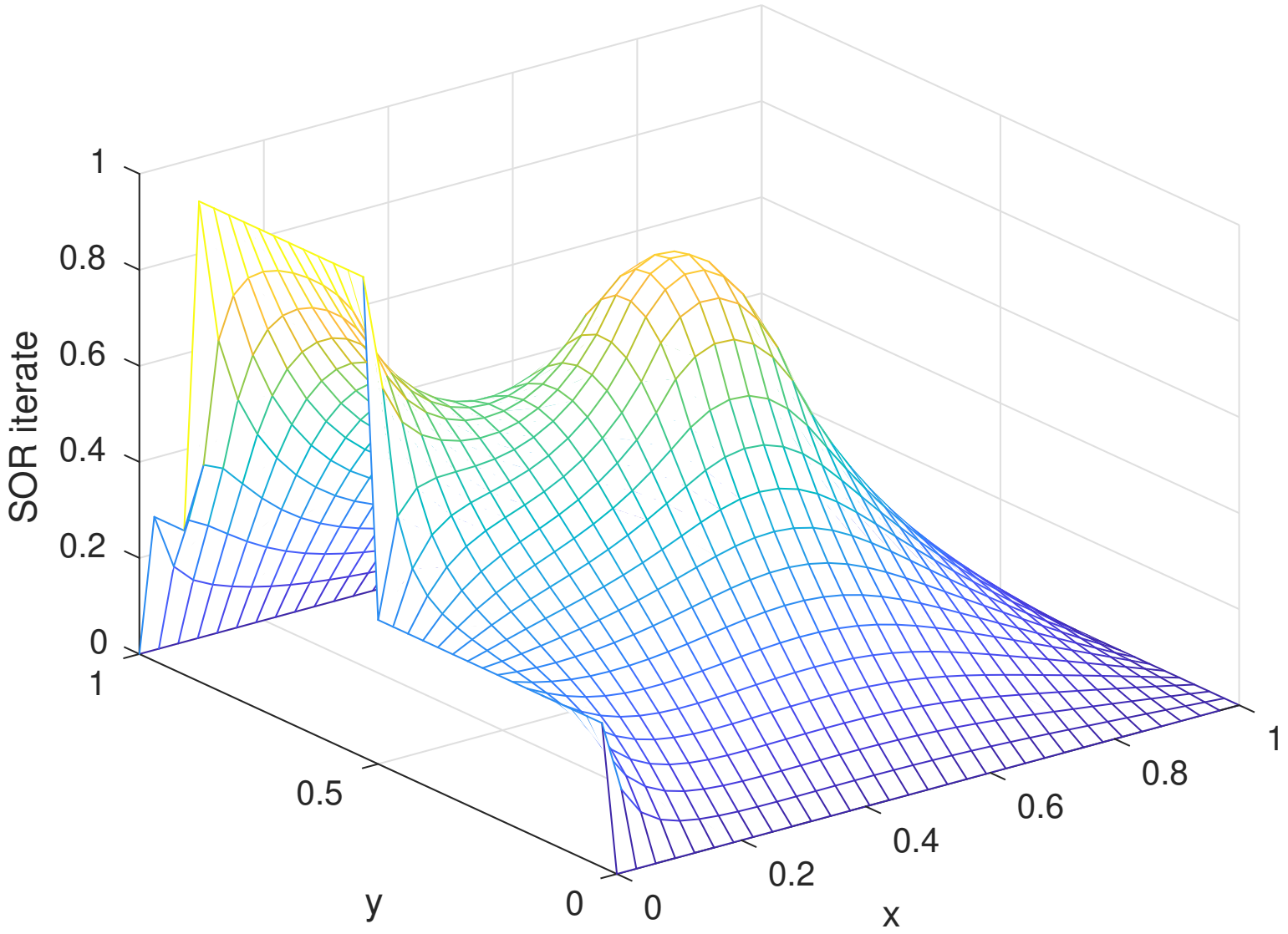
SOR iteration 24



SOR iteration 25



SOR iteration 26



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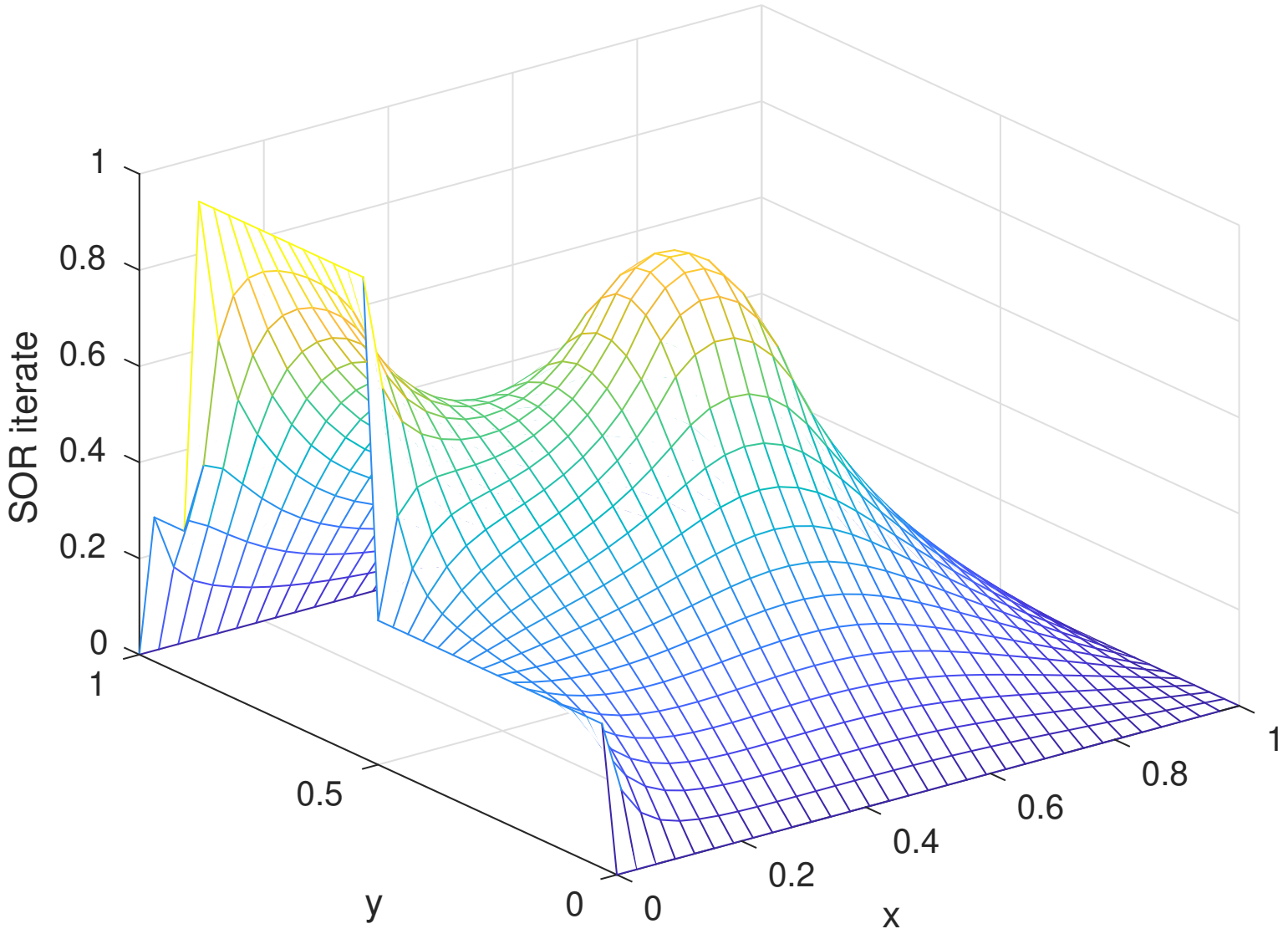
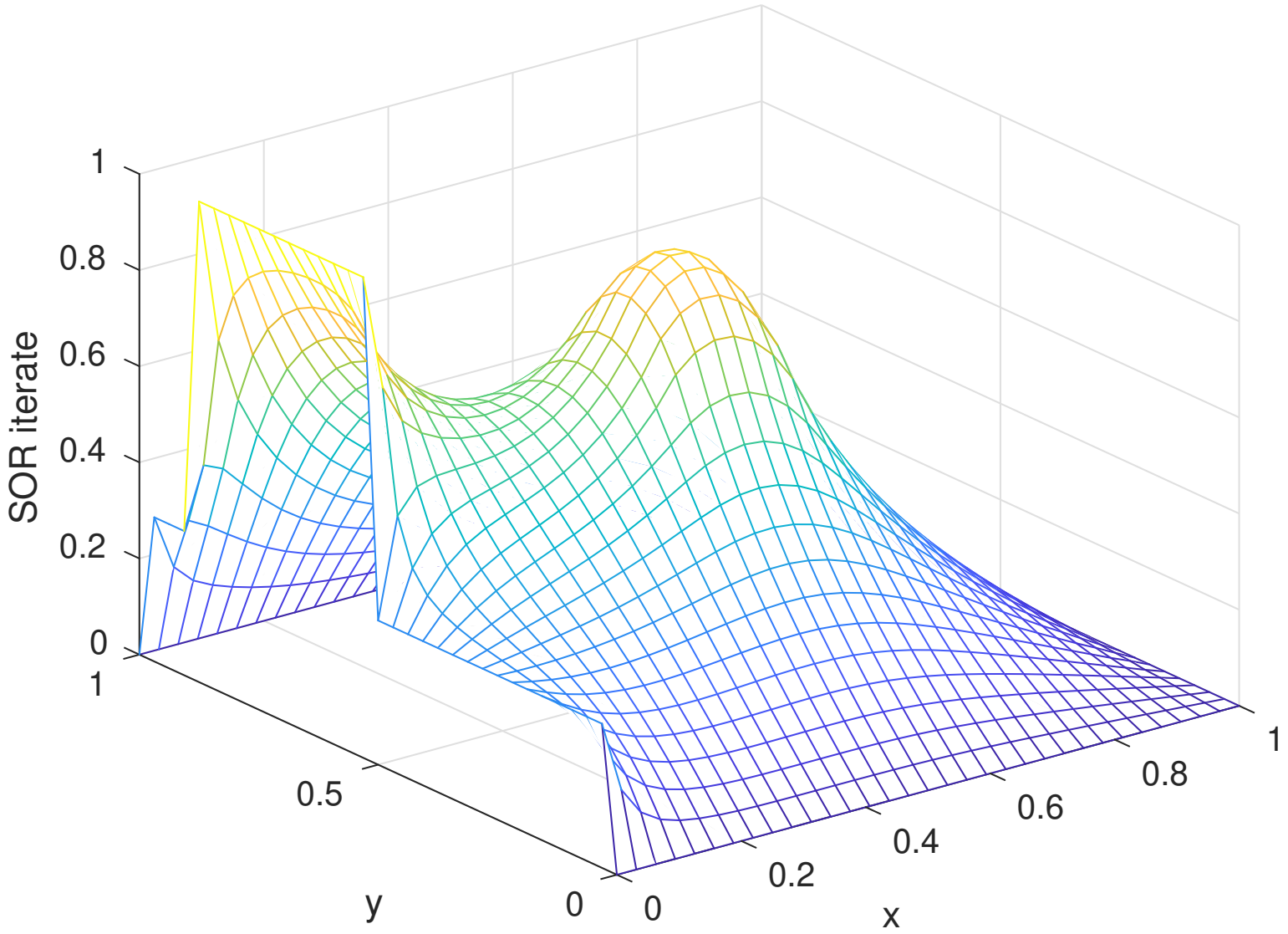
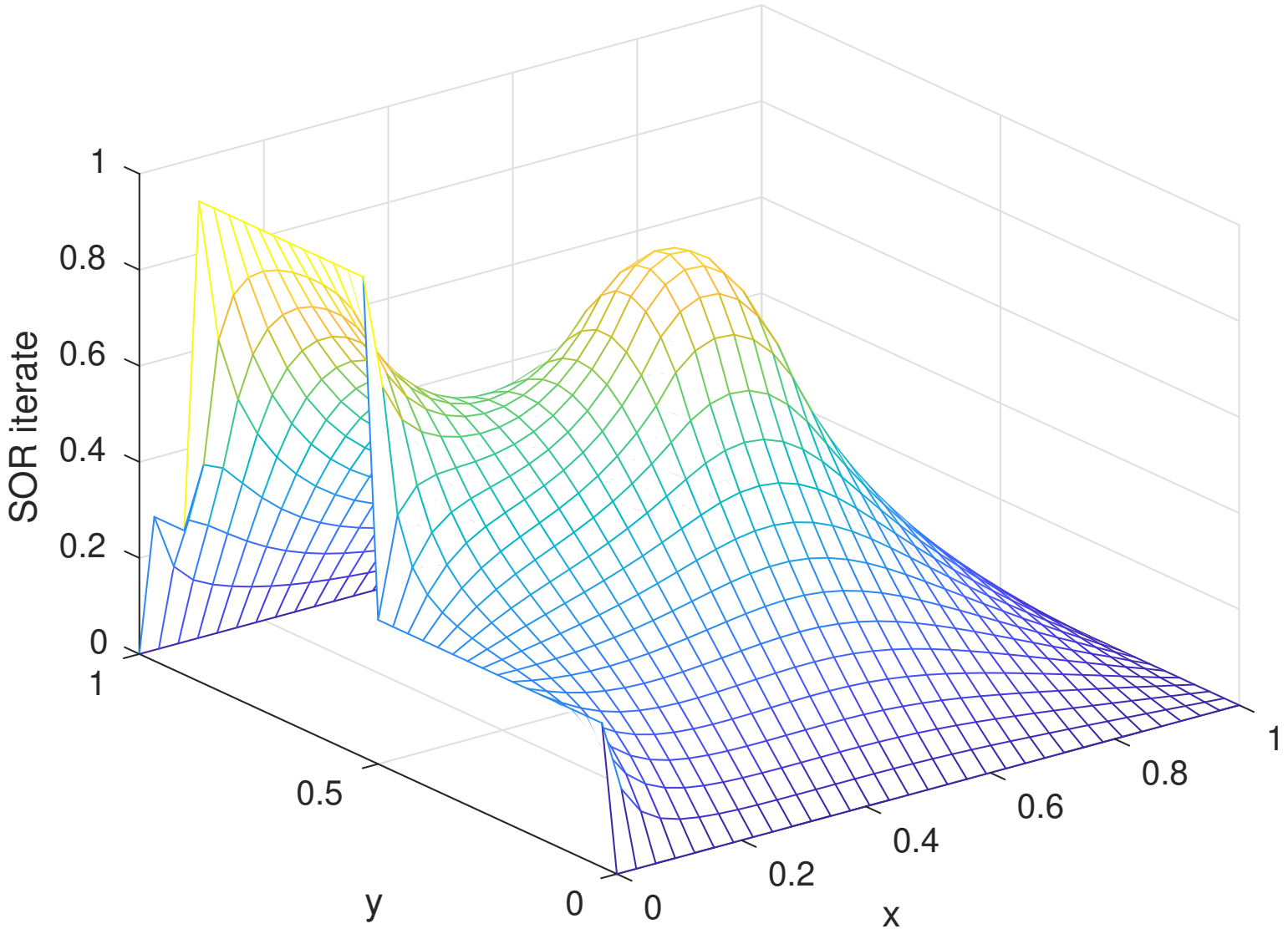
SOR iteration 27



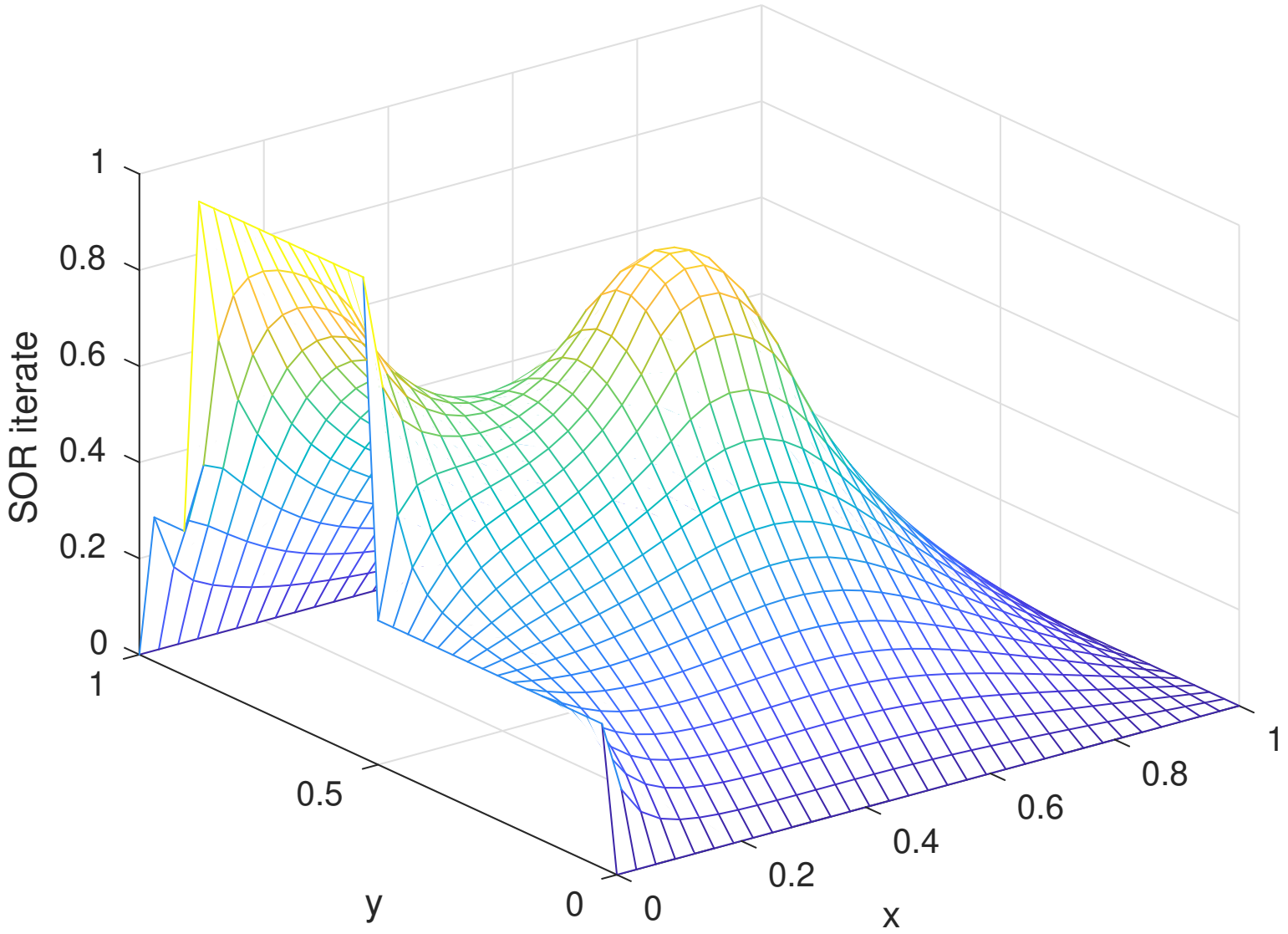
SOR iteration 28



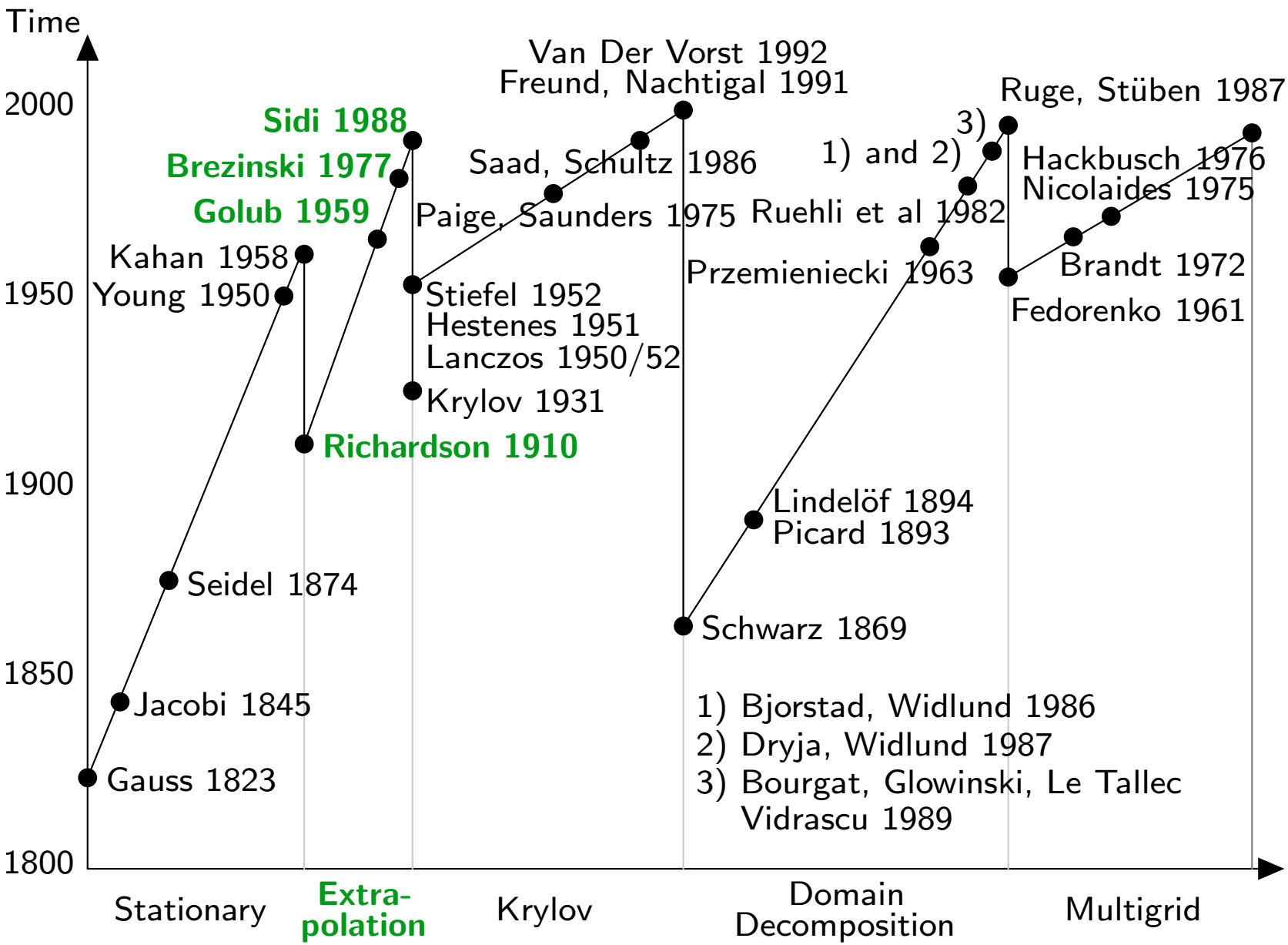
SOR iteration 29



SOR iteration 30



Extrapolation Methods for Linear Systems



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Richardson's Influential Paper

Richardson (1910): The Approximate Arithmetical Solution by Finite Differences of Physical Problems involving Differential Equations, with an Application to the stresses in a Masonry Dam

- *Historical section*: cites Runge, Heun, Kutta etc.
- Complete treatment of *finite difference methods*
- *Time stepping* for evolution problems
- *Richardson extrapolation*
- *Wall clock times and computational cost*: "about $\frac{n}{18}$ pence per co-ordinate point, n being the number of digits [...] quickest boys averaged 2000 relaxations of Δ_h per week with 3 digits".
- *Richardson's iterative method* (simply $M = I$)

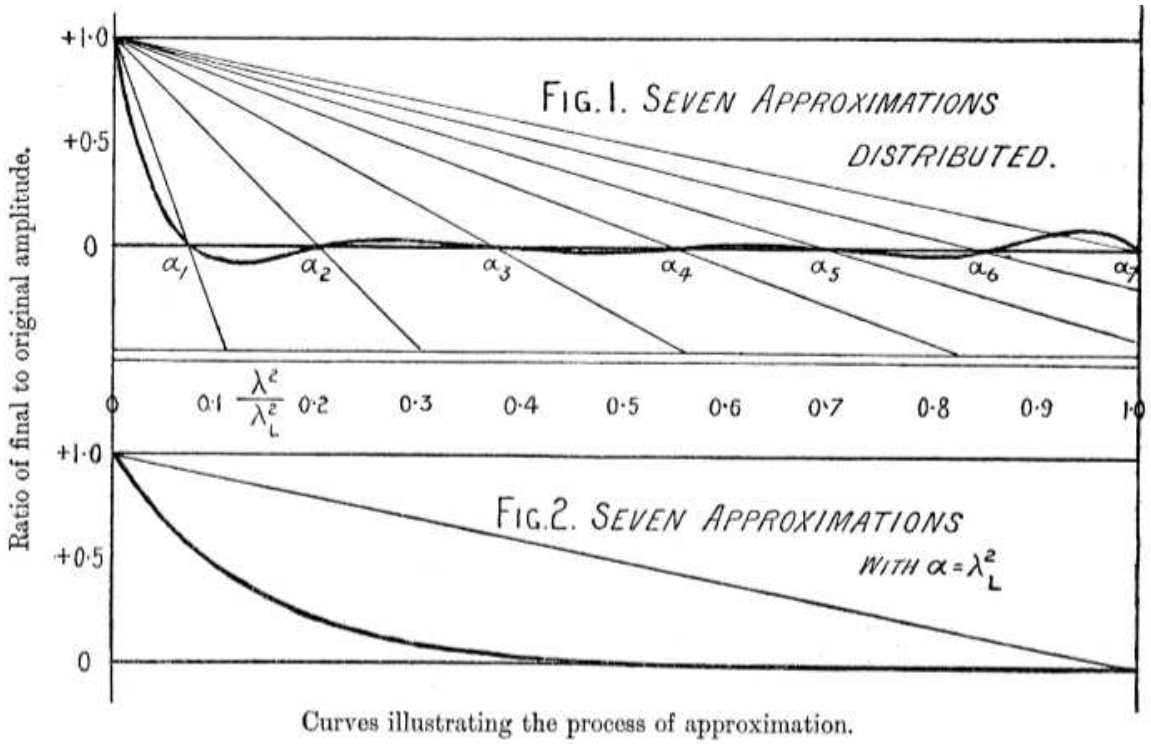
$$\boldsymbol{u}^{n+1} = \boldsymbol{u}^n + \alpha_n(\boldsymbol{f} - A\boldsymbol{u}^n).$$

- *Solution of the masonry dam problem.*

Richardson's Optimization of α_n

$$\mathrm{SI}\,(\phi_{t+1}-\phi_u)^2 = \Sigma A_k^2 \left[\left(1-\frac{\lambda_k^2}{\alpha_1}\right)\left(1-\frac{\lambda_k^2}{\alpha_2}\right)\times\ldots\times\left(1-\frac{\lambda_k^2}{\alpha_t}\right)\right]^2 \quad . \quad . \quad . \quad (9).$$

Now it has been found that by a judicious choice of $\alpha_1, \alpha_2, \ldots, \alpha_t$, the quantity $\left[\left(1-\frac{\lambda_k^2}{\alpha_1}\right)\left(1-\frac{\lambda_k^2}{\alpha_2}\right)\times\ldots\times\left(1-\frac{\lambda_k^2}{\alpha_t}\right)\right]^2$ may be made small for all possible values of λ_k^2. (Thus fig. 1 shows this done for a set of seven (α)'s. This graph was arrived at by trial.) The error E_{t+1} of ϕ_{t+1} may therefore be made small in comparison with that



Curves illustrating the process of approximation.

Chebyshev (1854): Théorie des mécanismes connus sous le nom de parallélogrammes.

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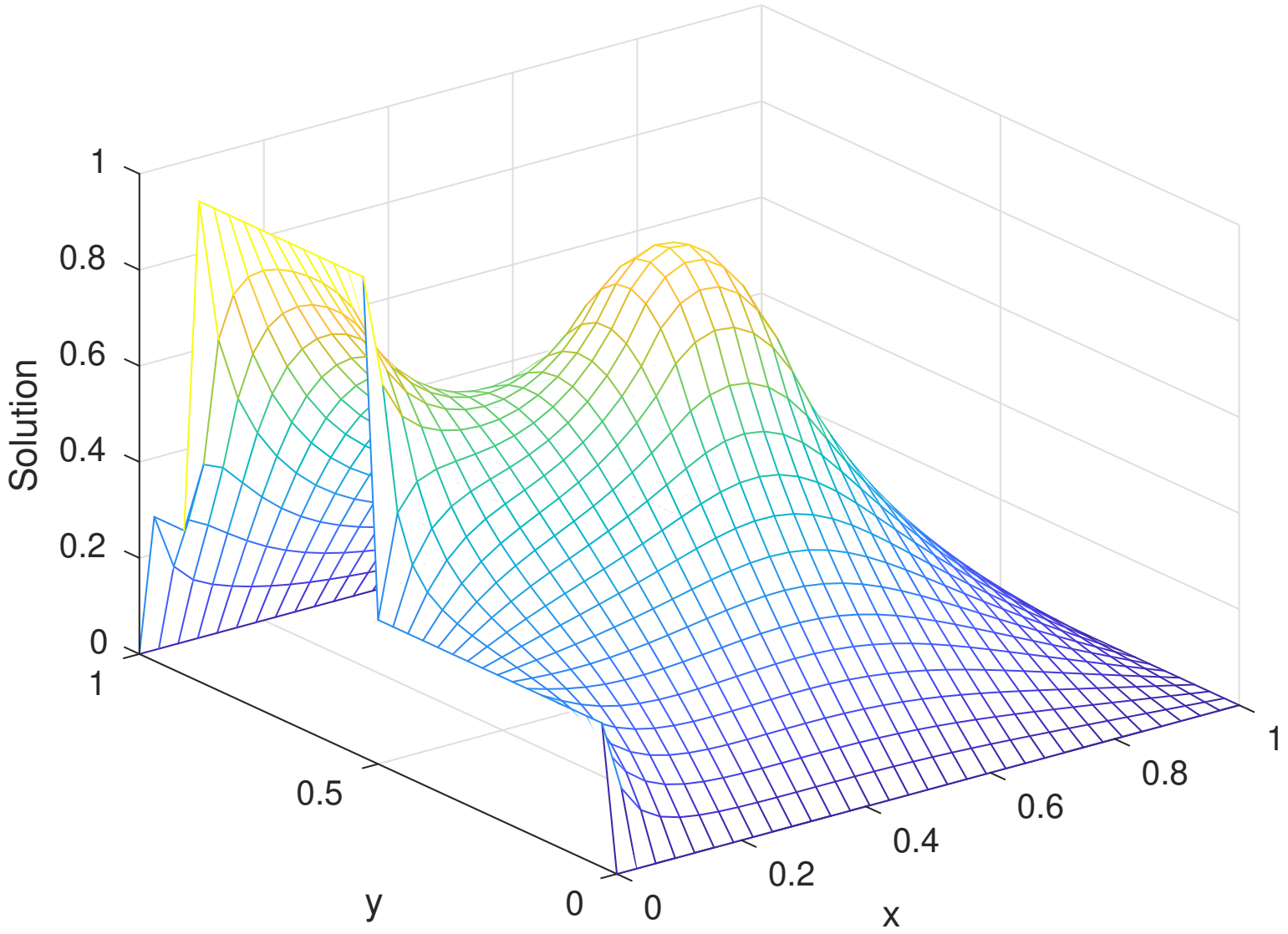
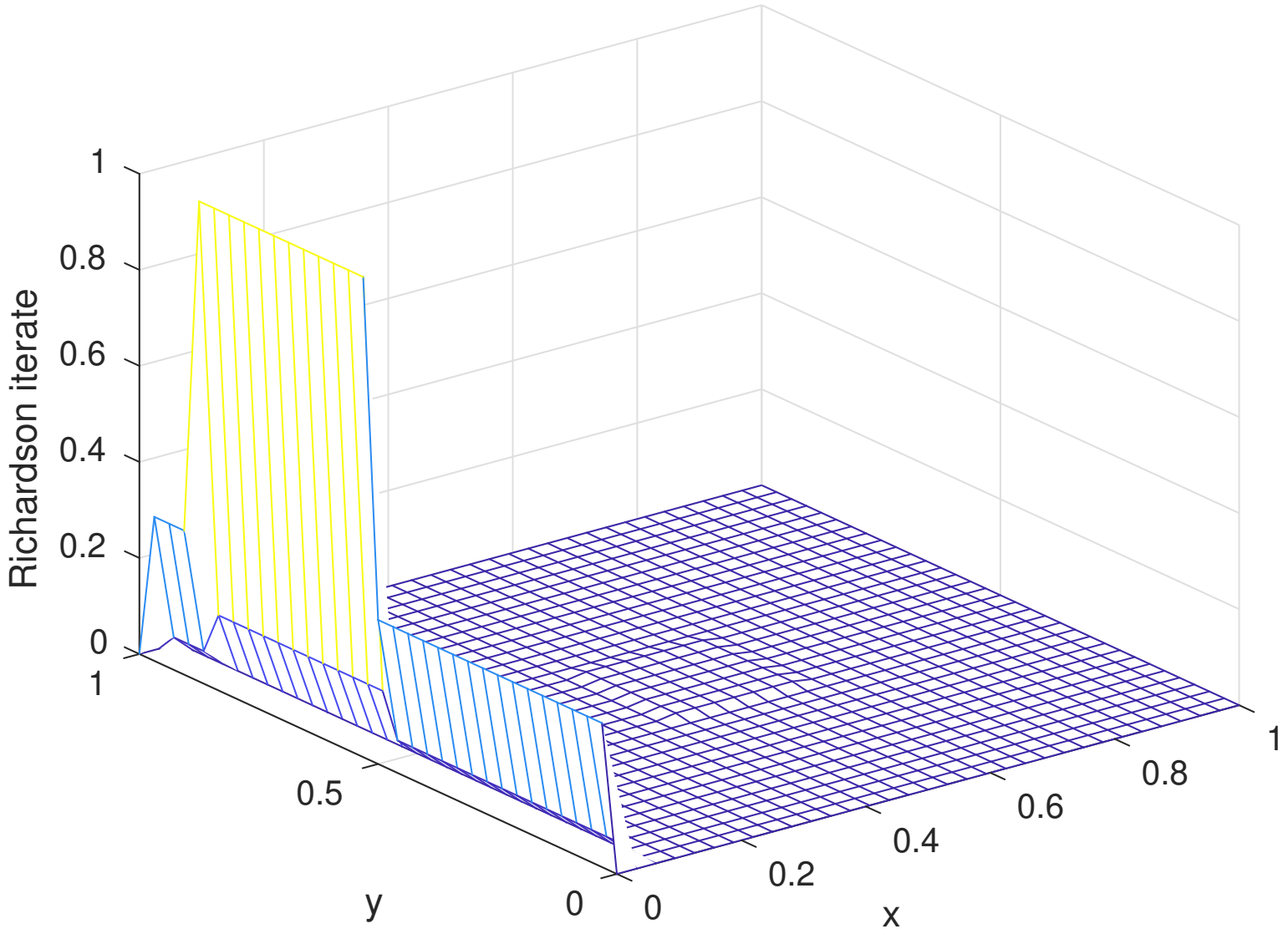
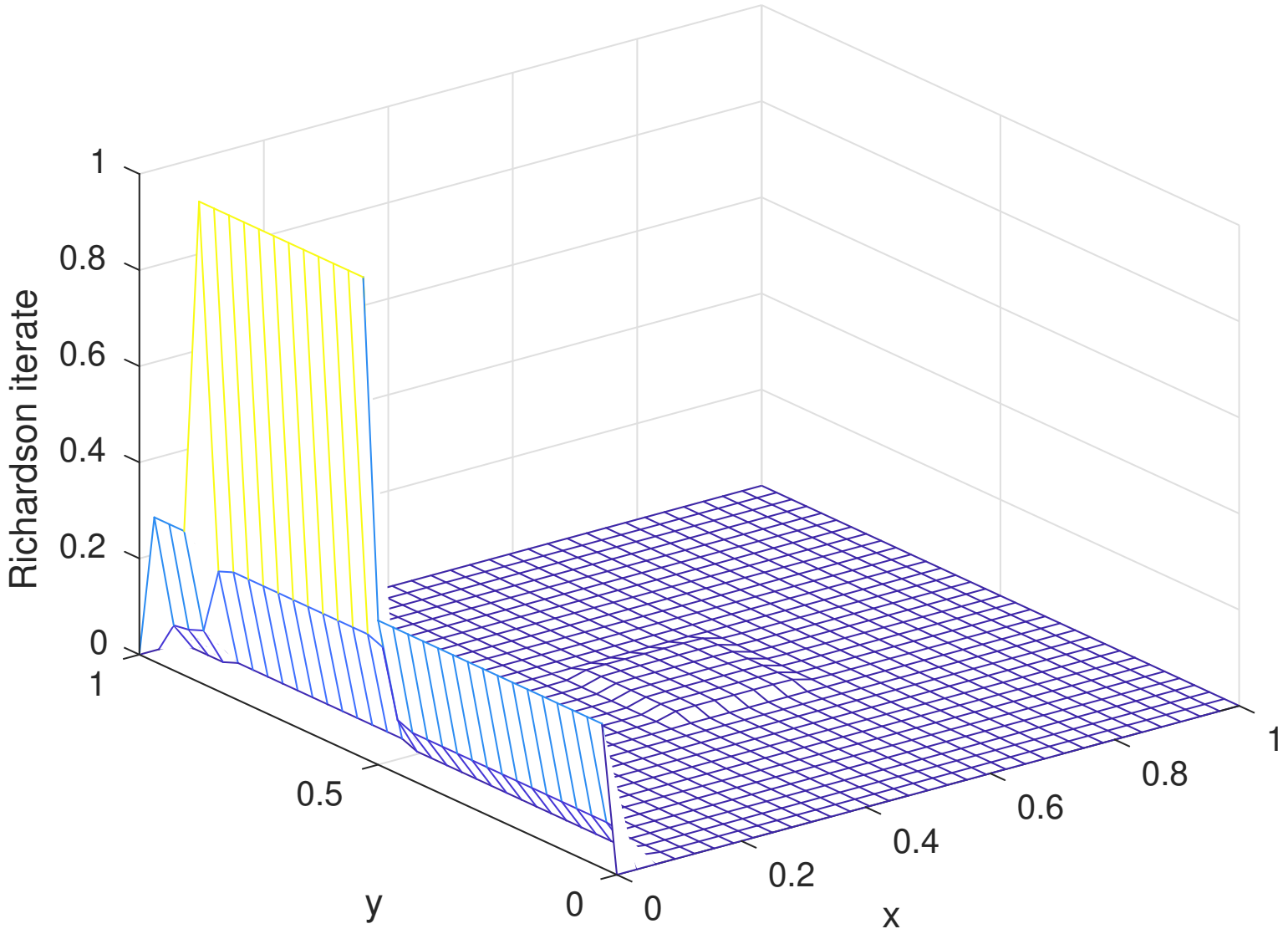
Example: temperature in a room



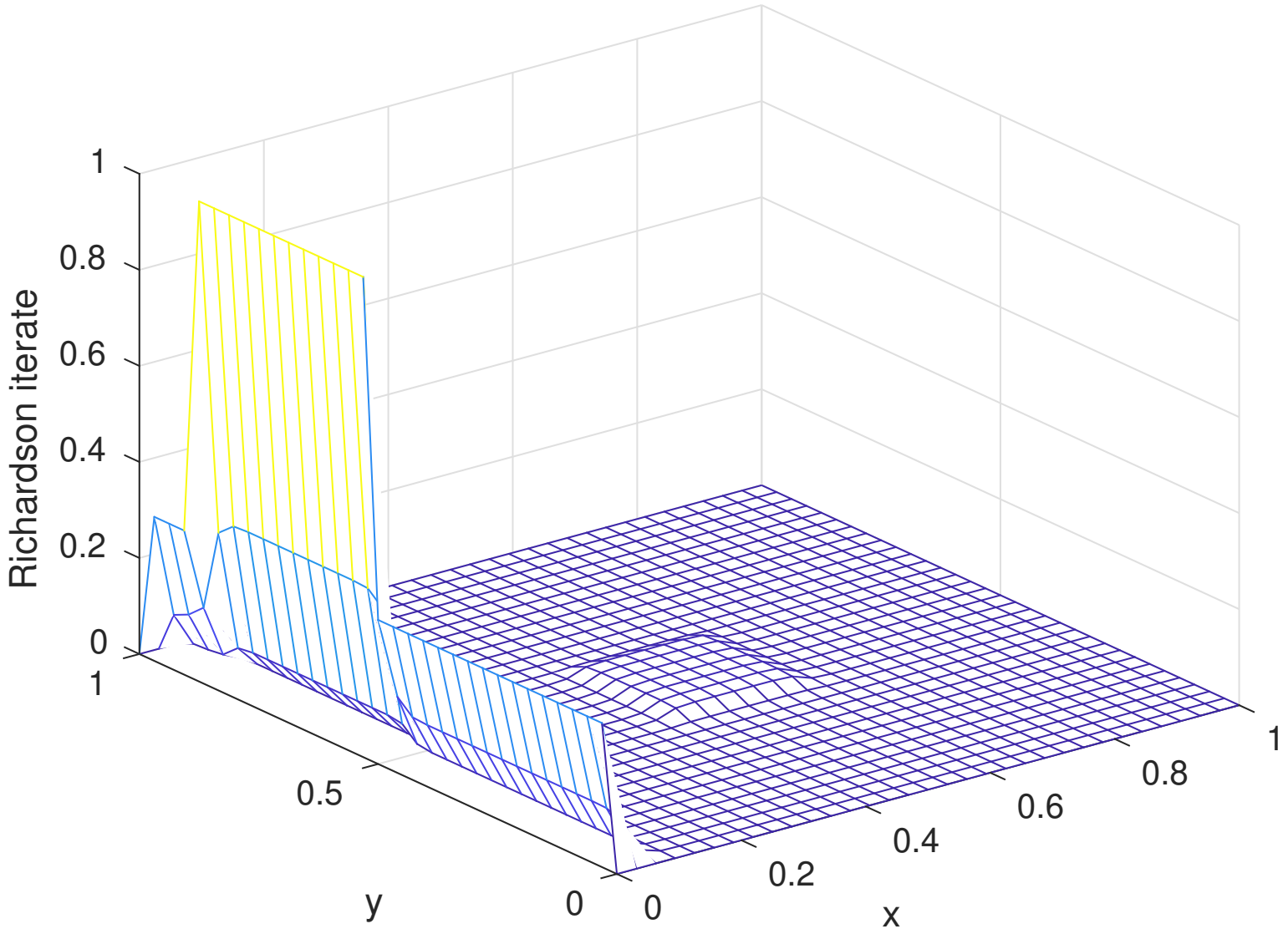
Richardson iteration 1



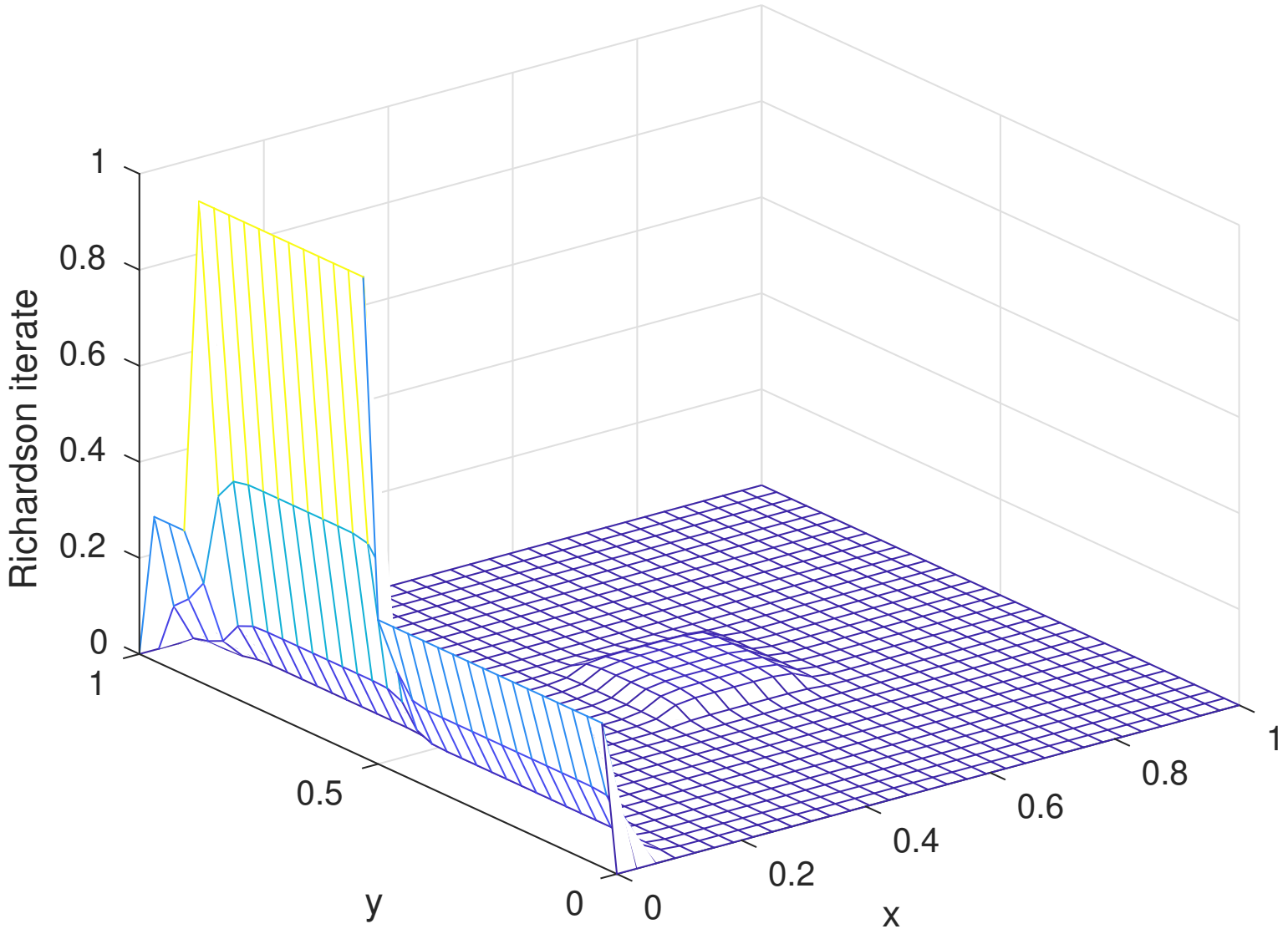
Richardson iteration 2



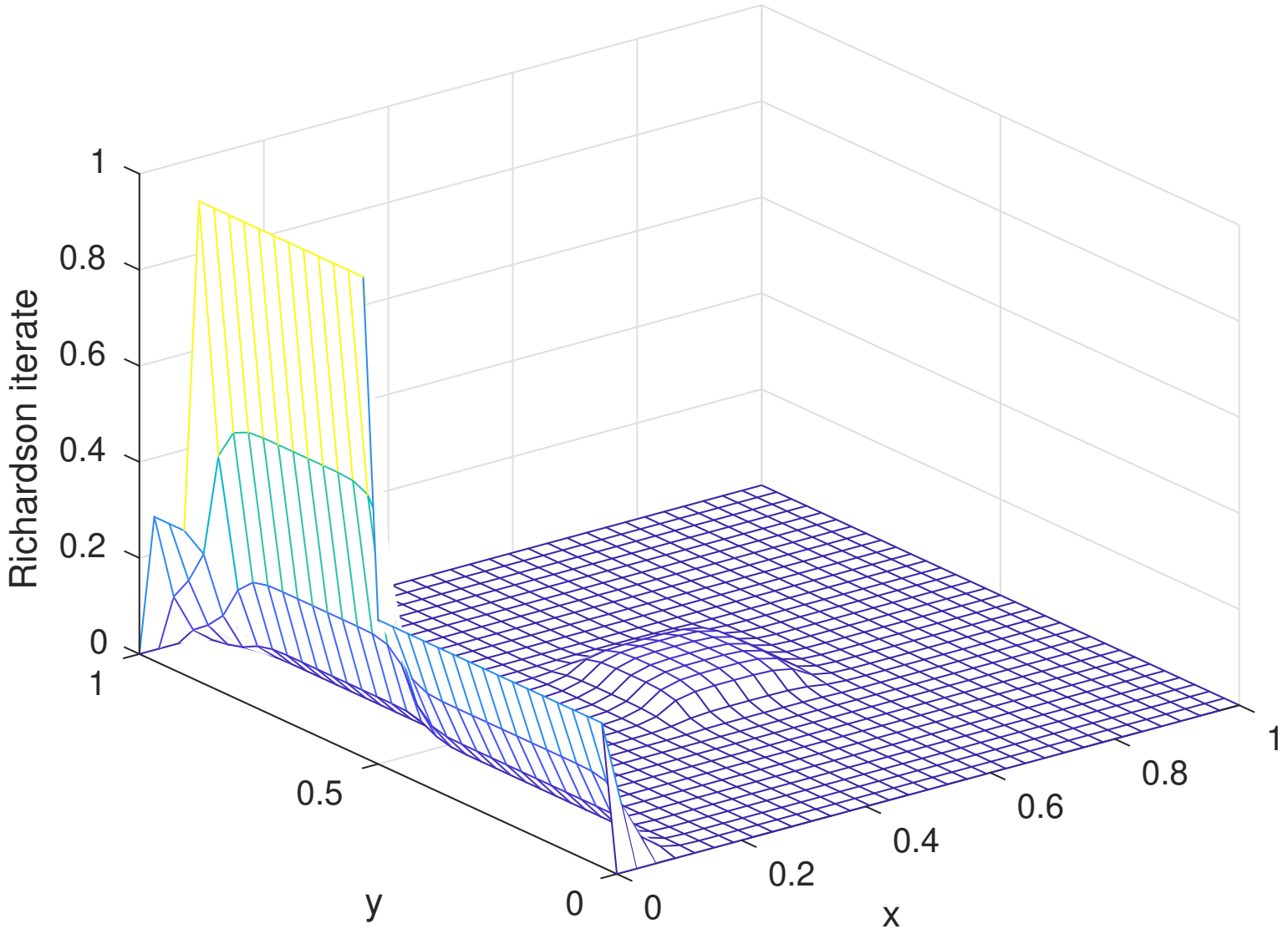
Richardson iteration 3



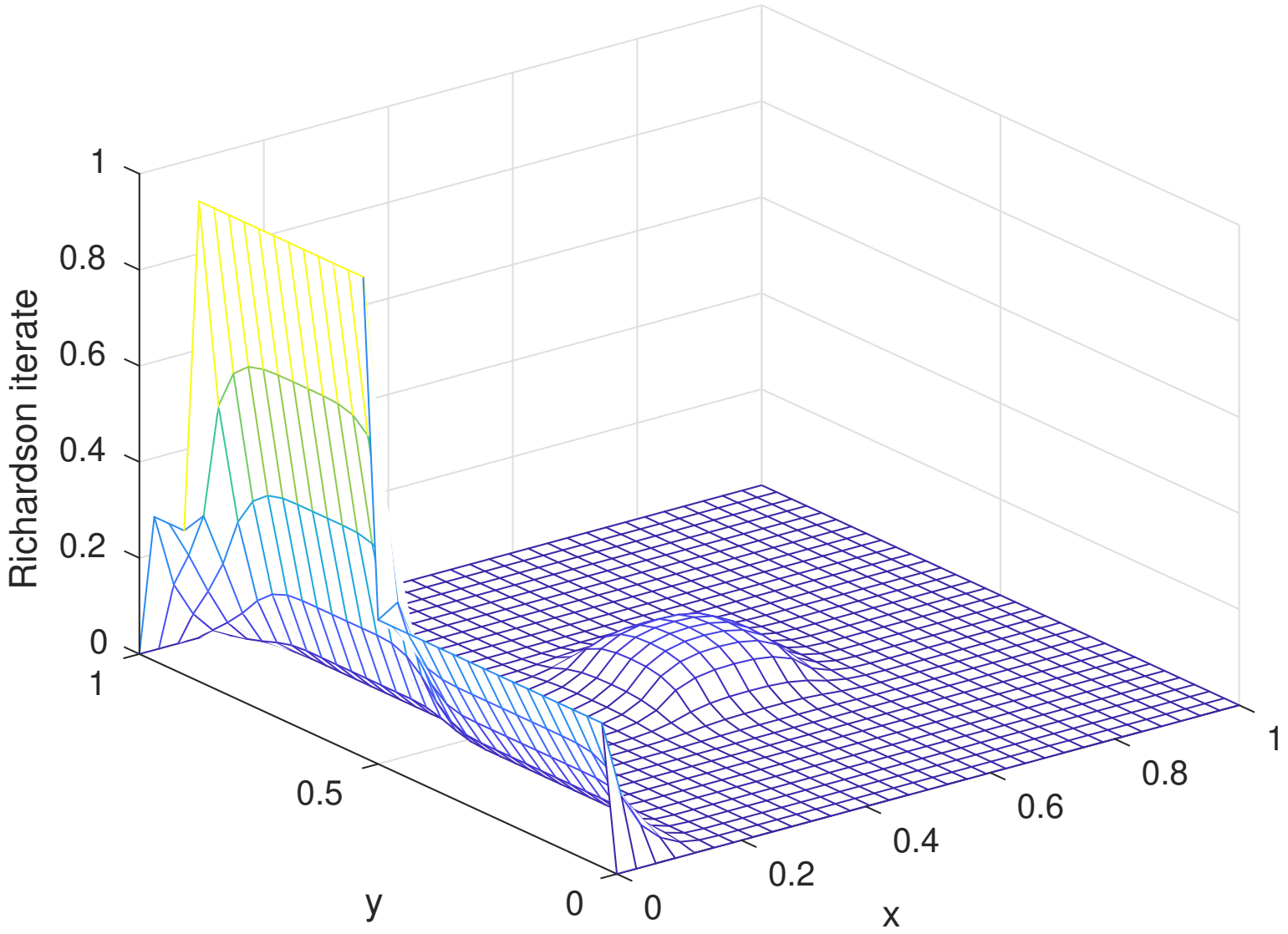
Richardson iteration 4



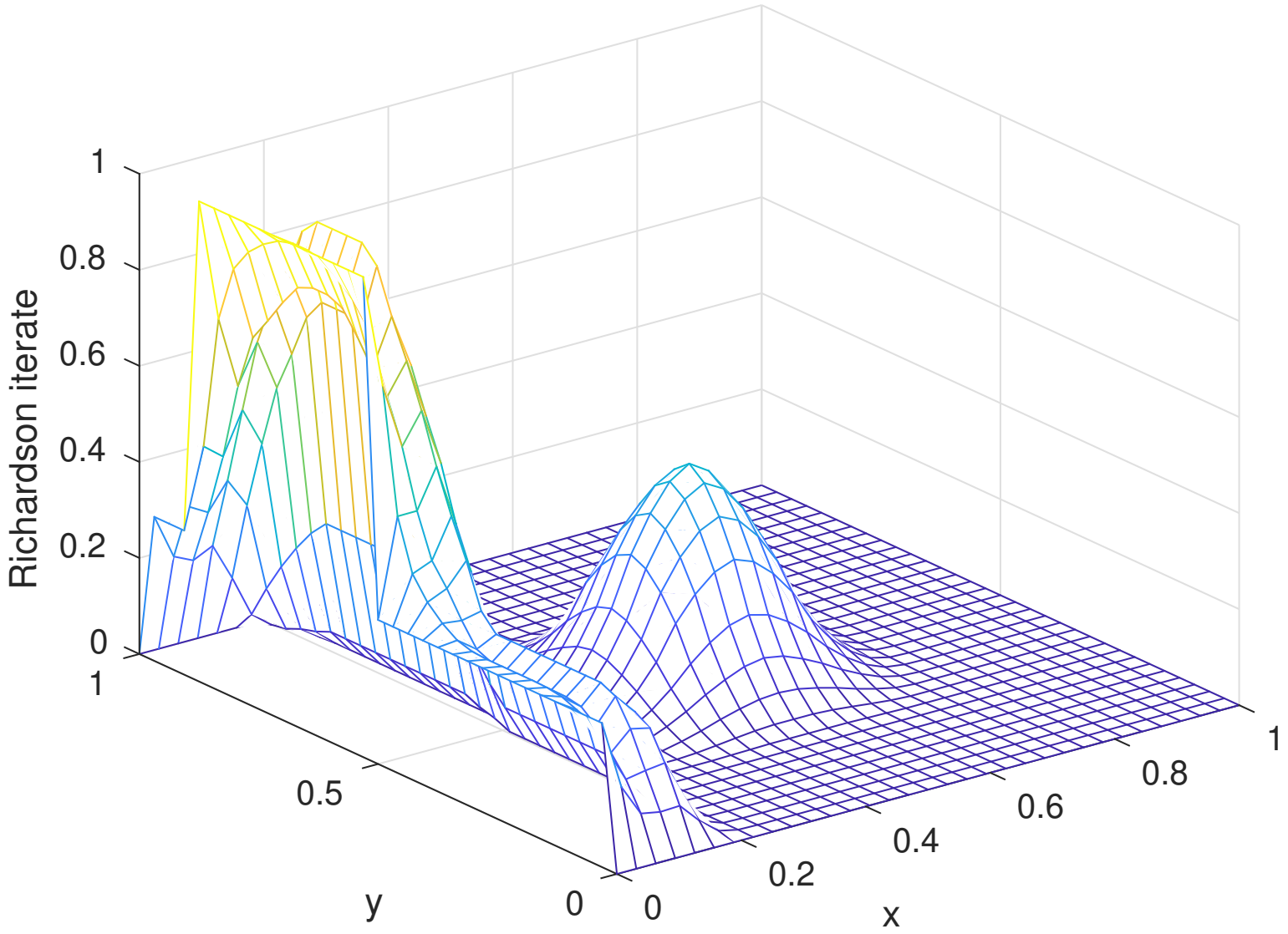
Richardson iteration 5



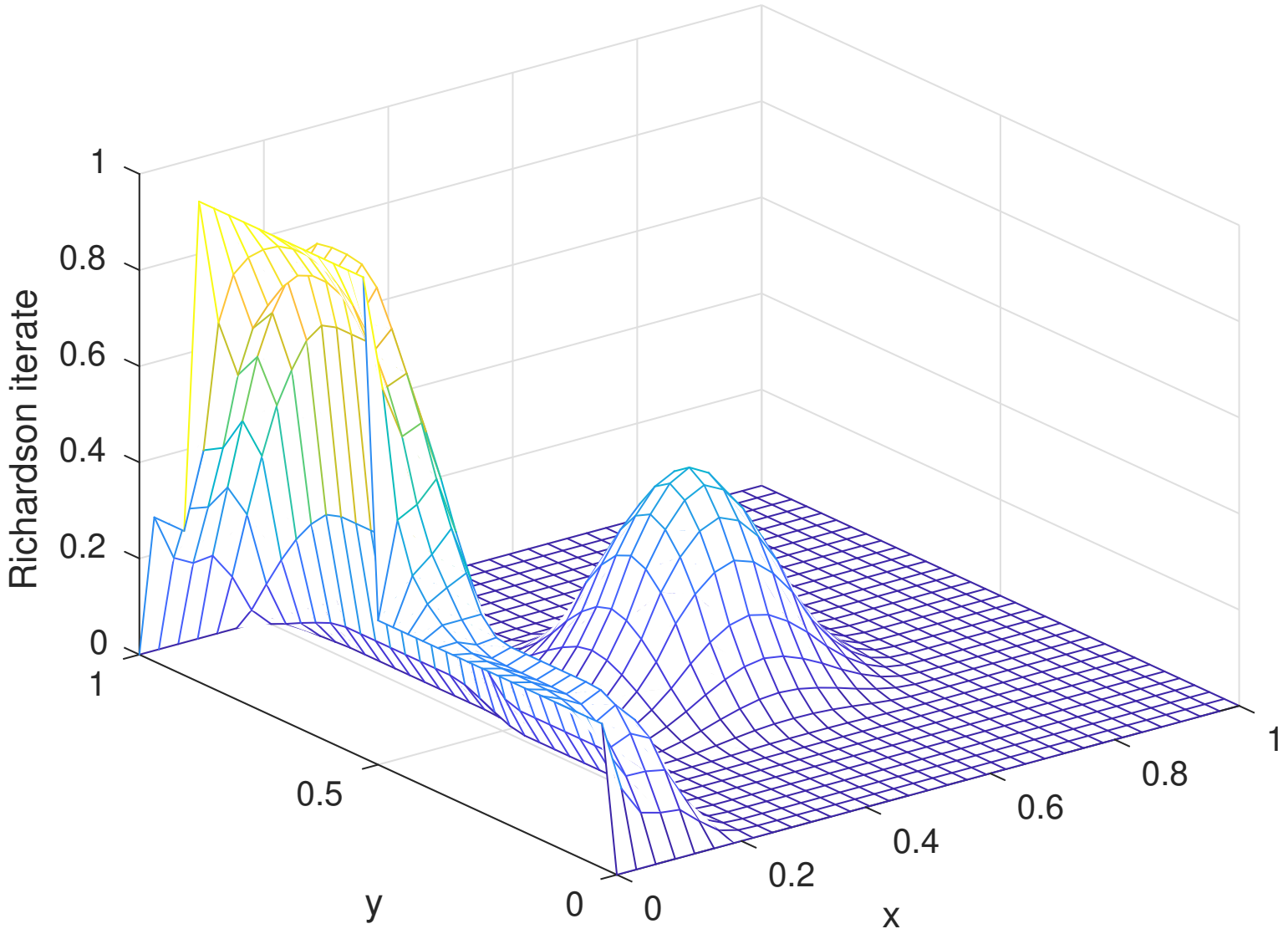
Richardson iteration 6



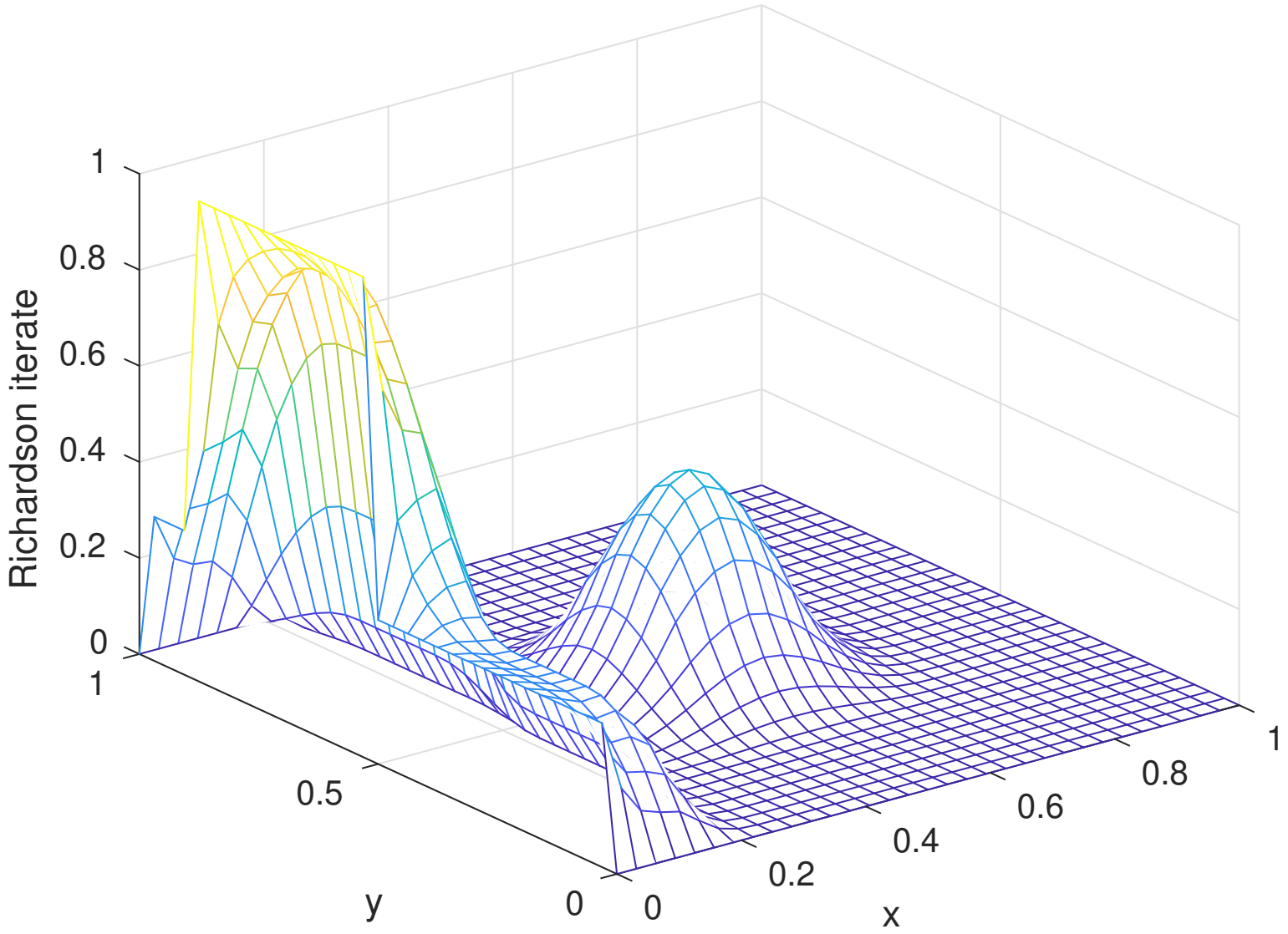
Richardson iteration 7



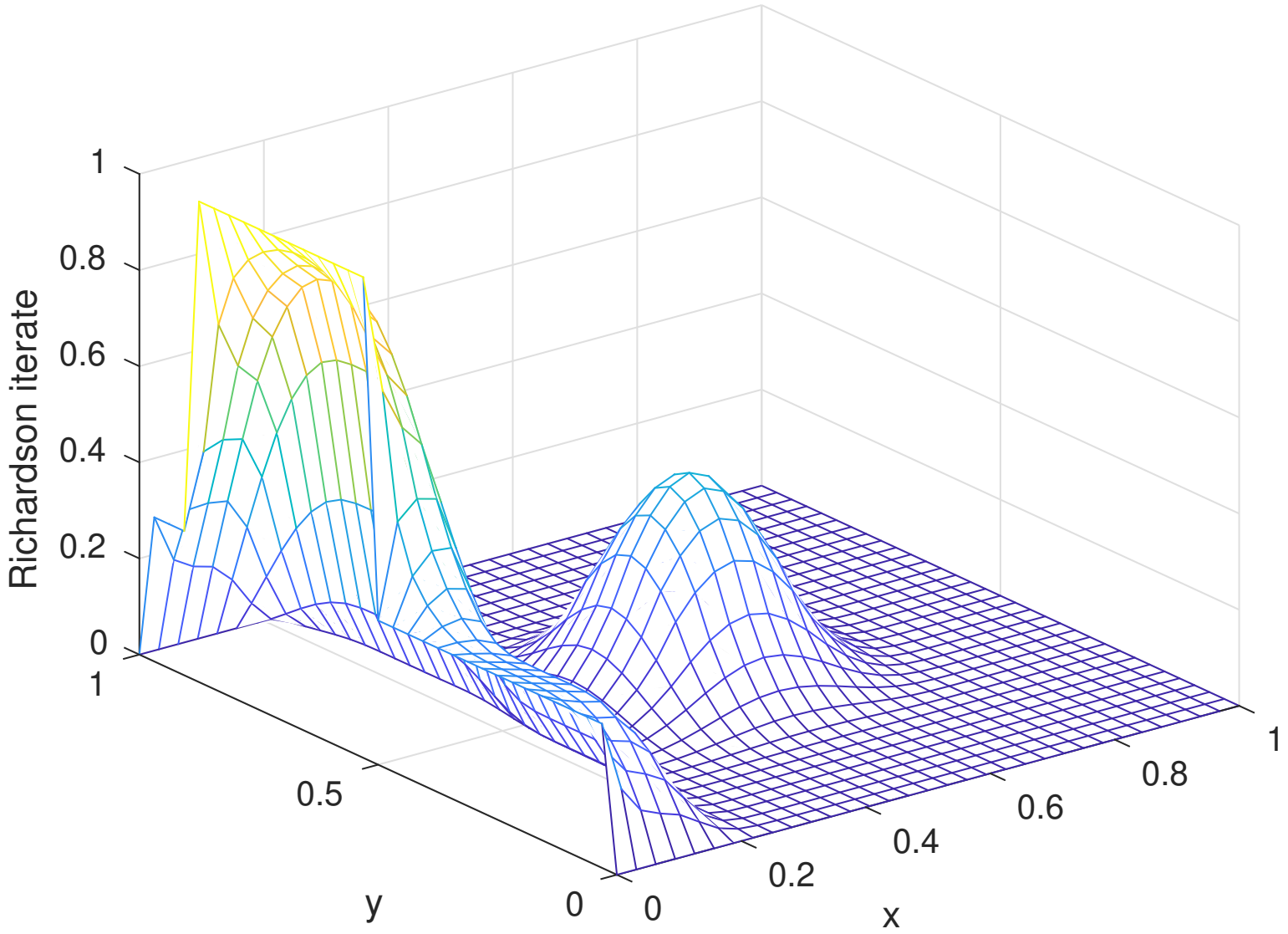
Richardson iteration 8



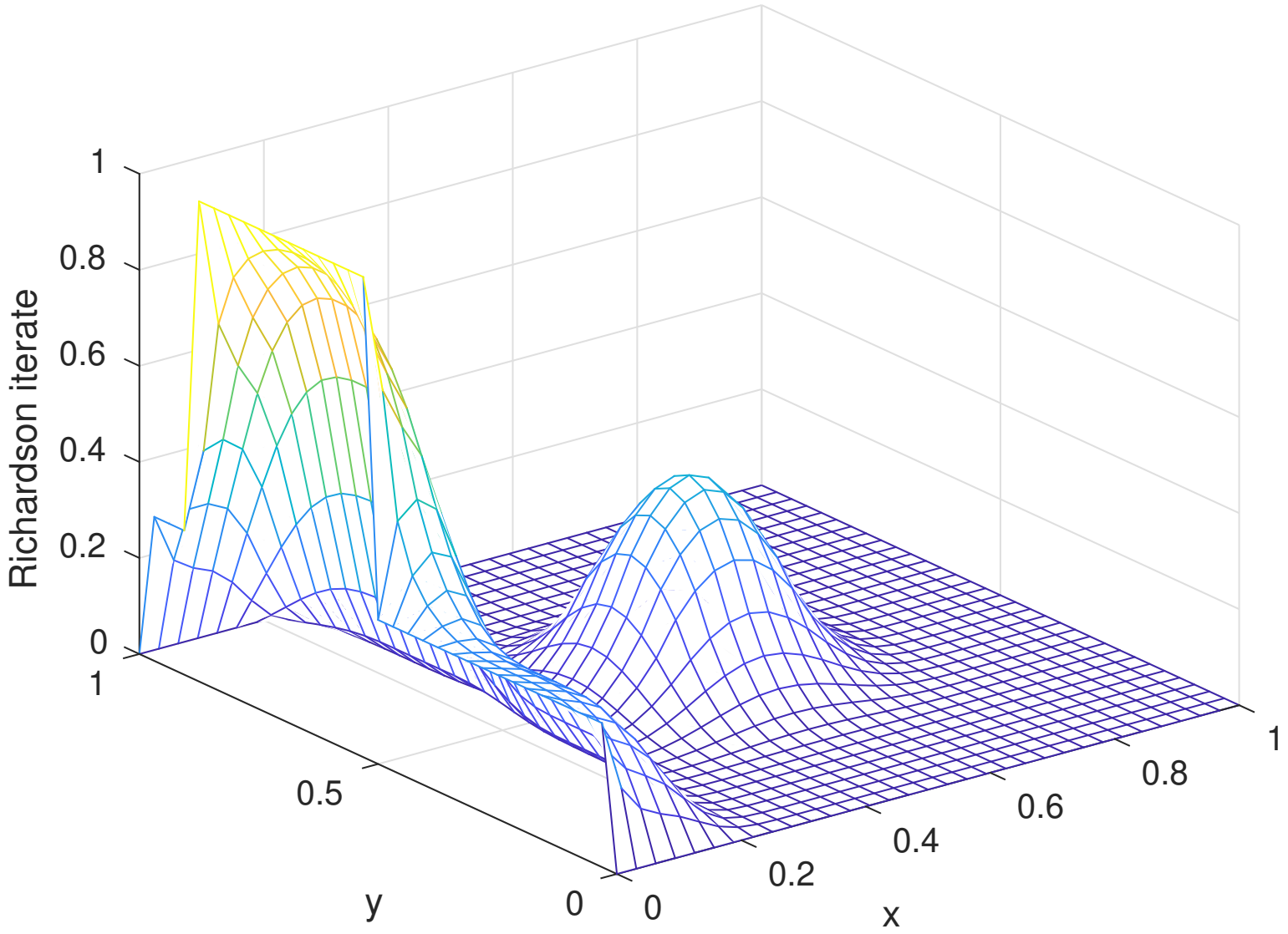
Richardson iteration 9



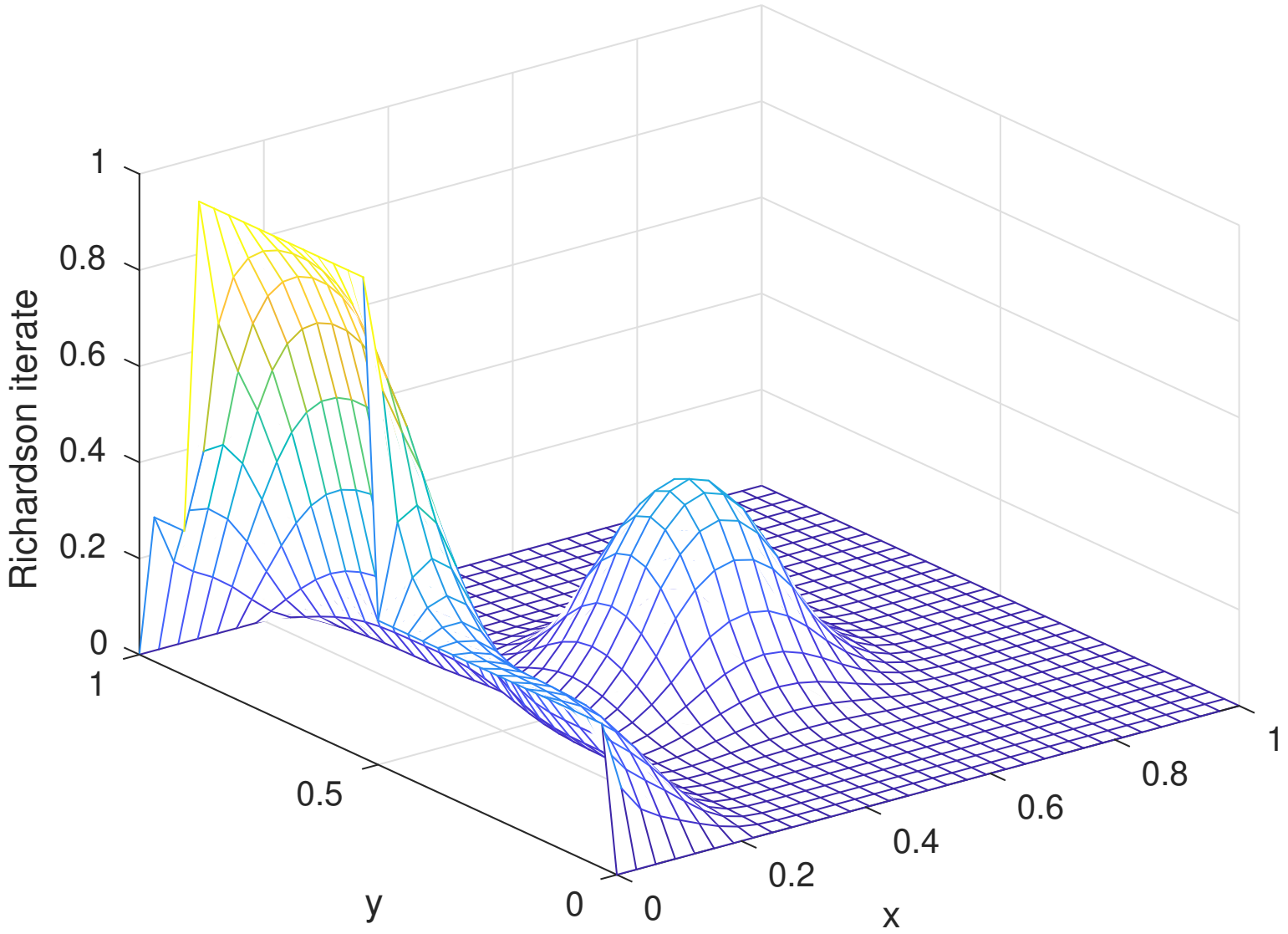
Richardson iteration 10



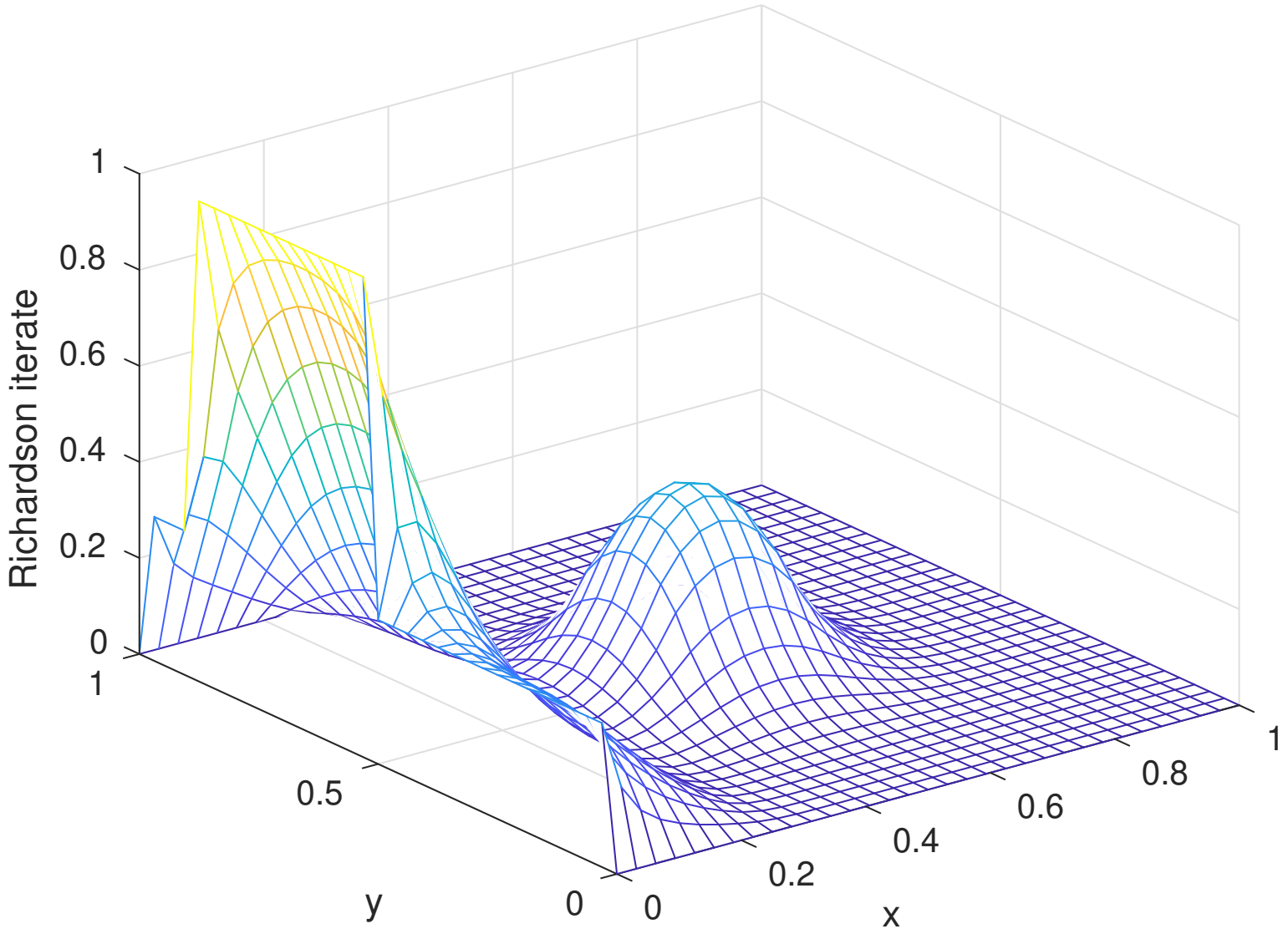
Richardson iteration 11



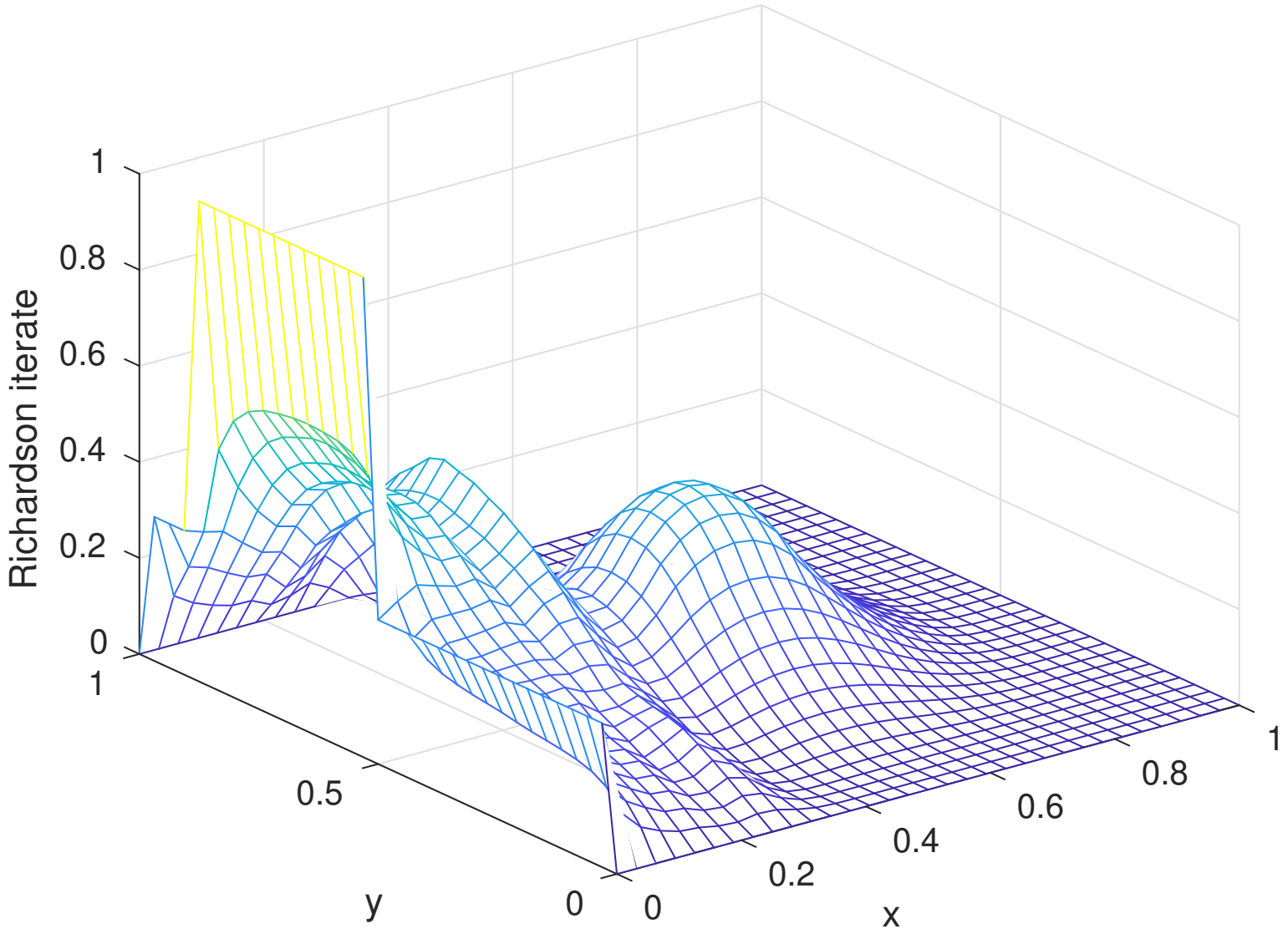
Richardson iteration 12



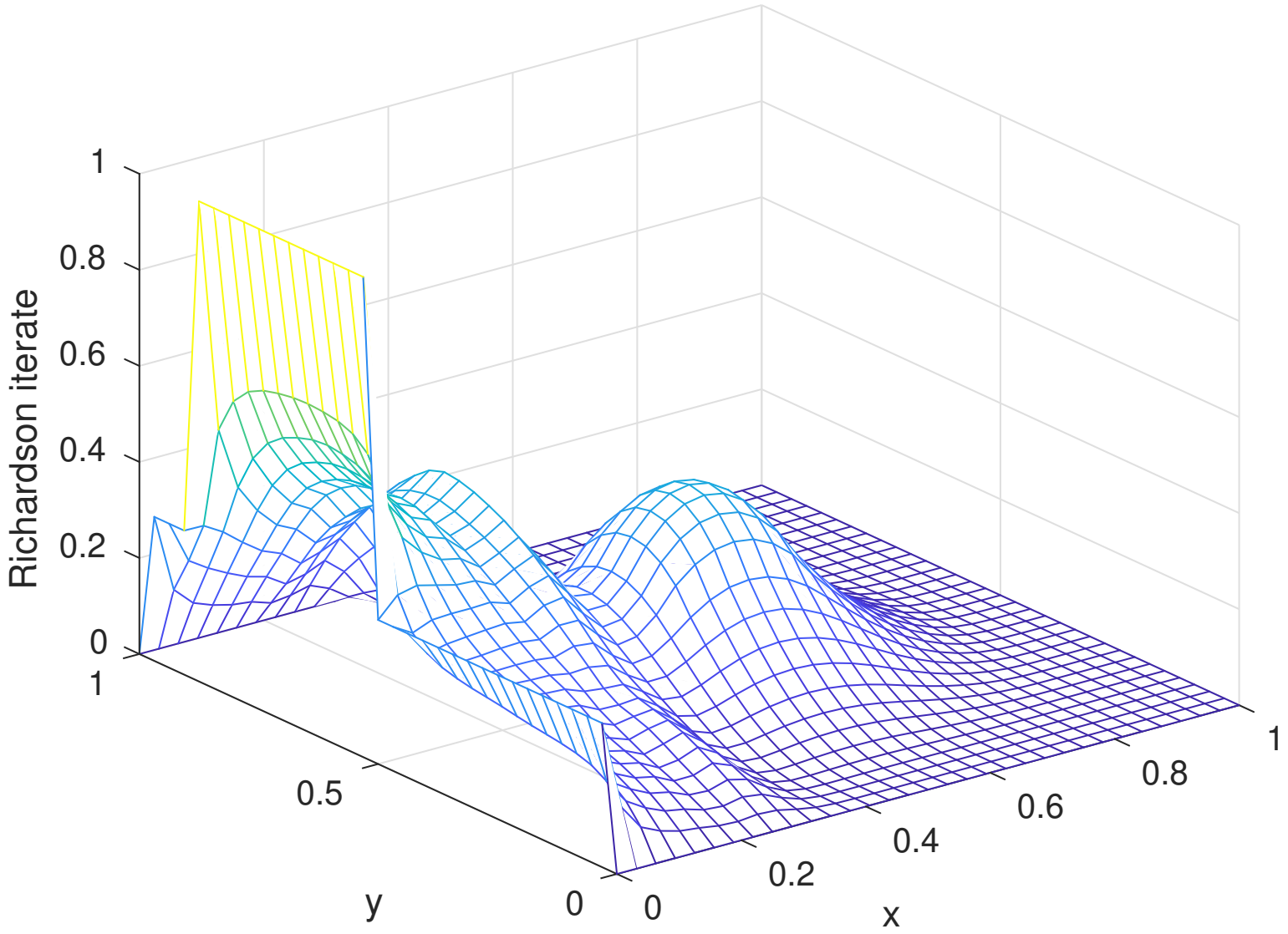
Richardson iteration 13



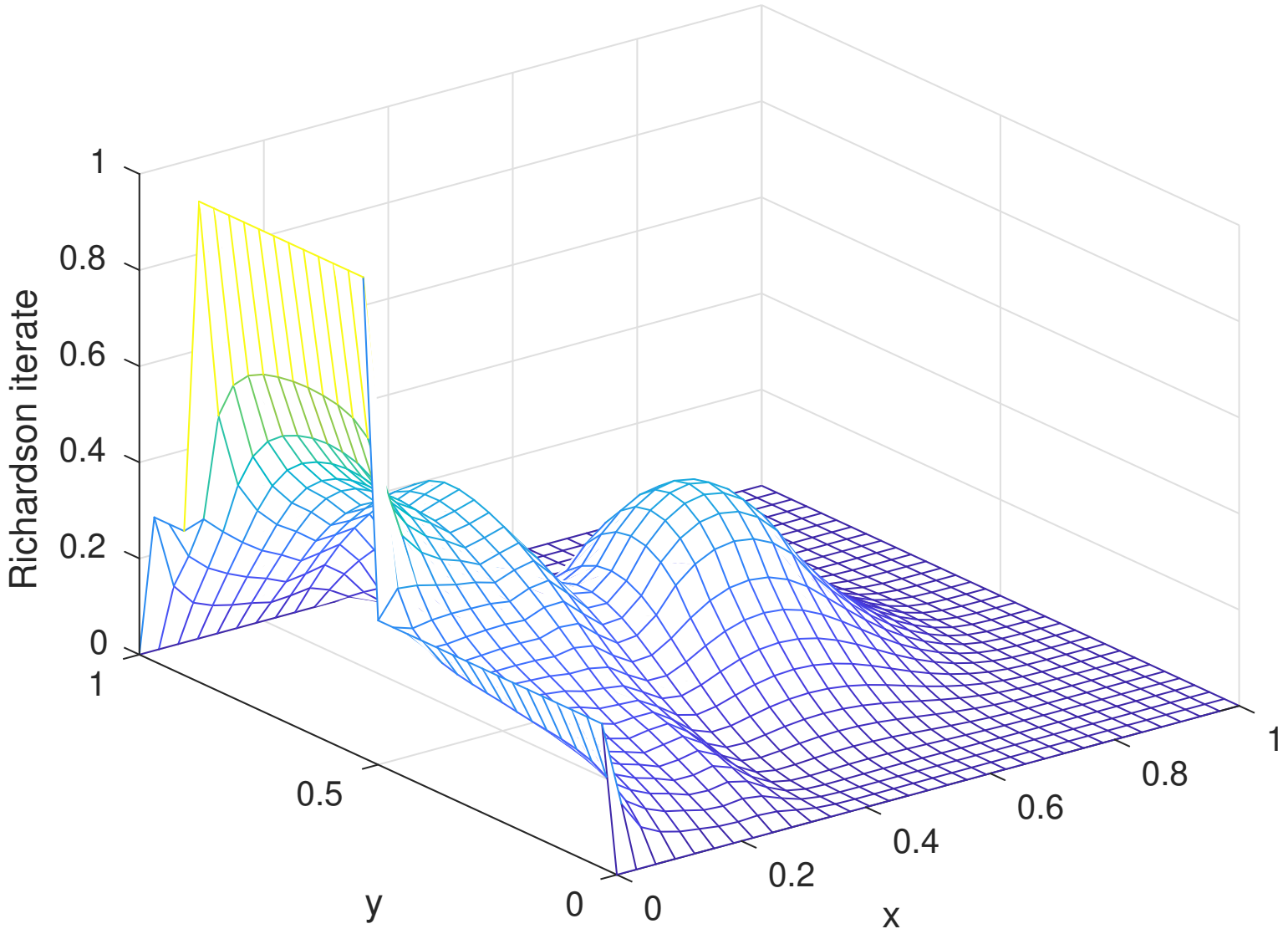
Richardson iteration 14



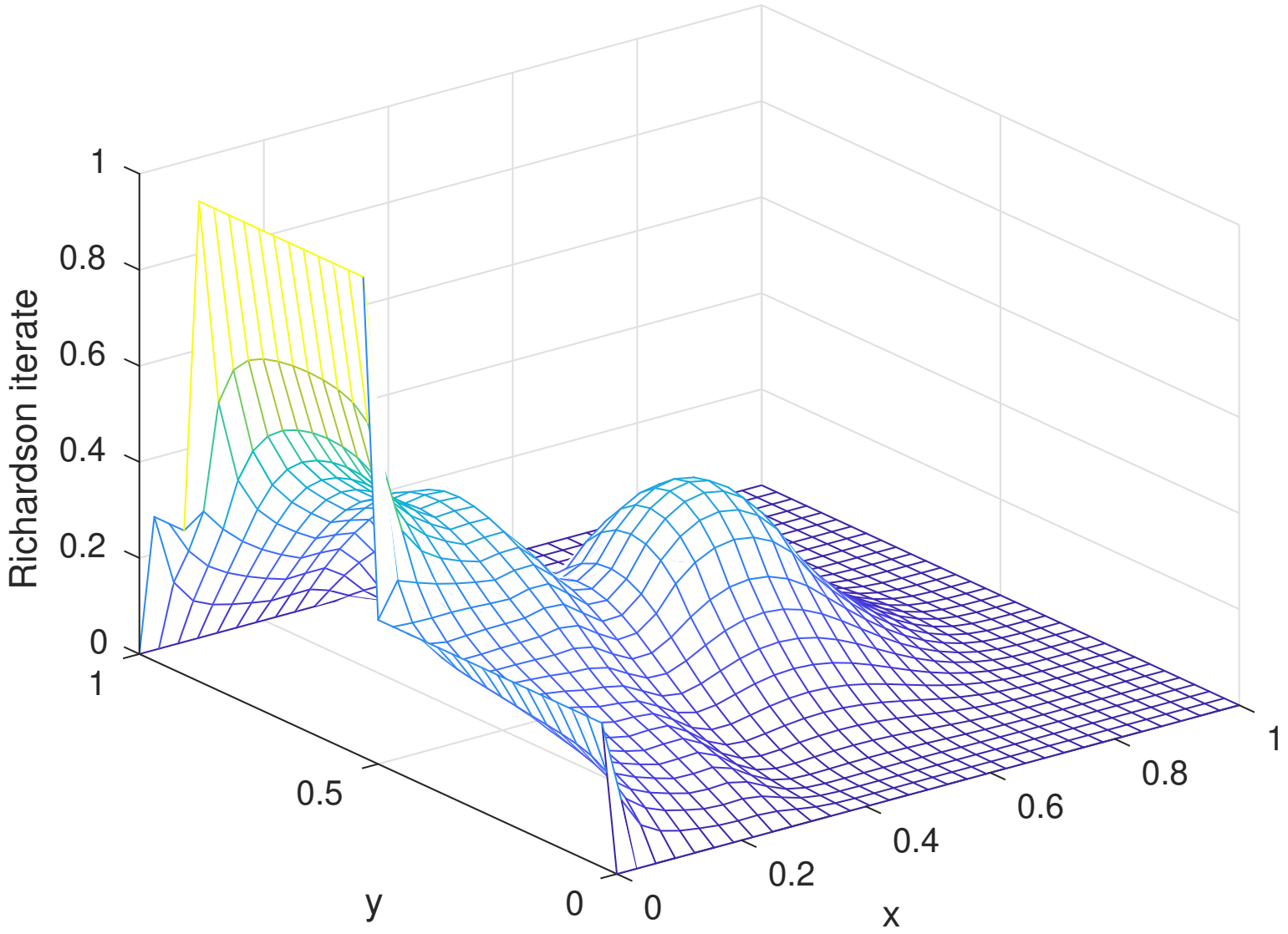
Richardson iteration 15



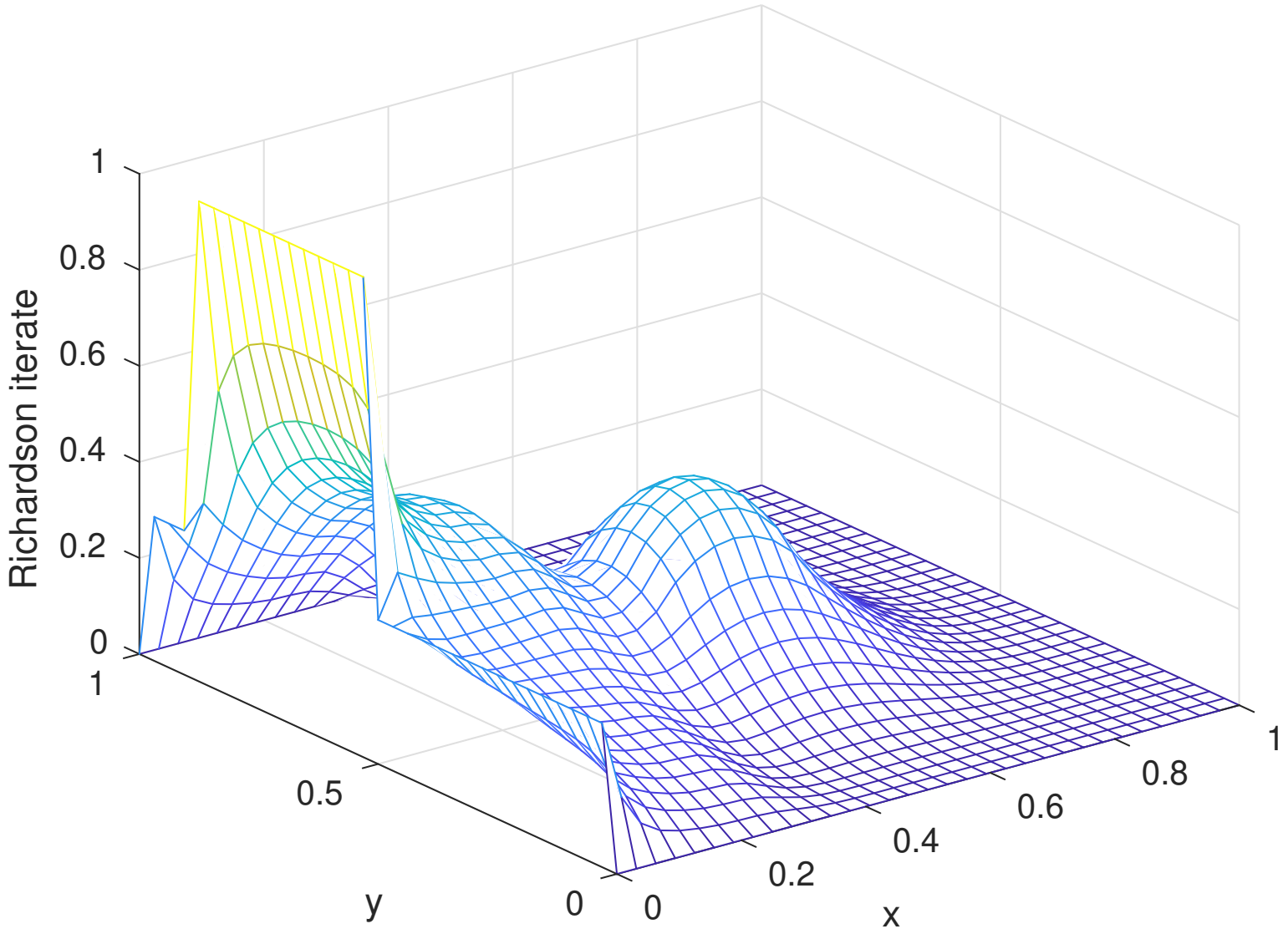
Richardson iteration 16



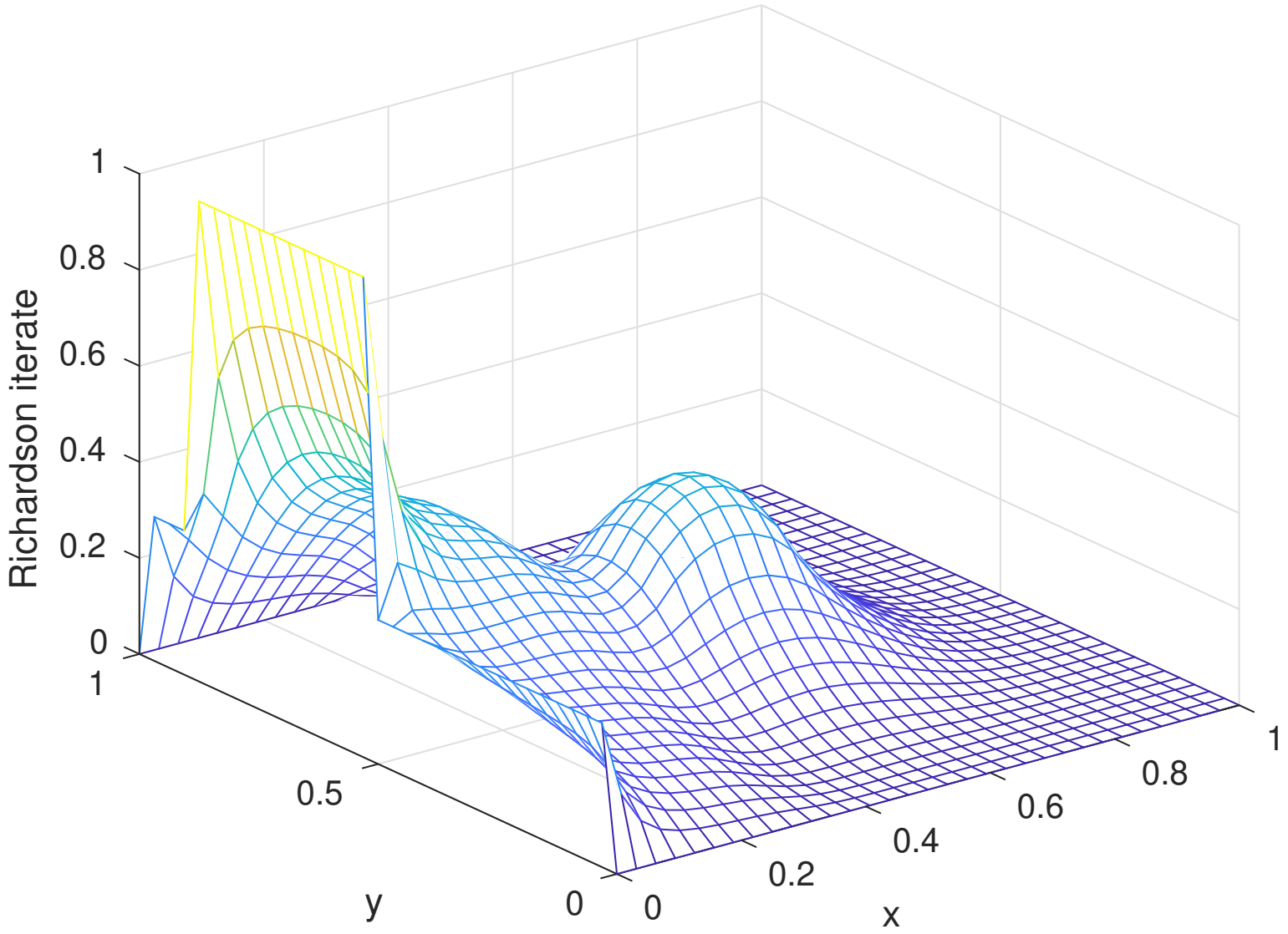
Richardson iteration 17



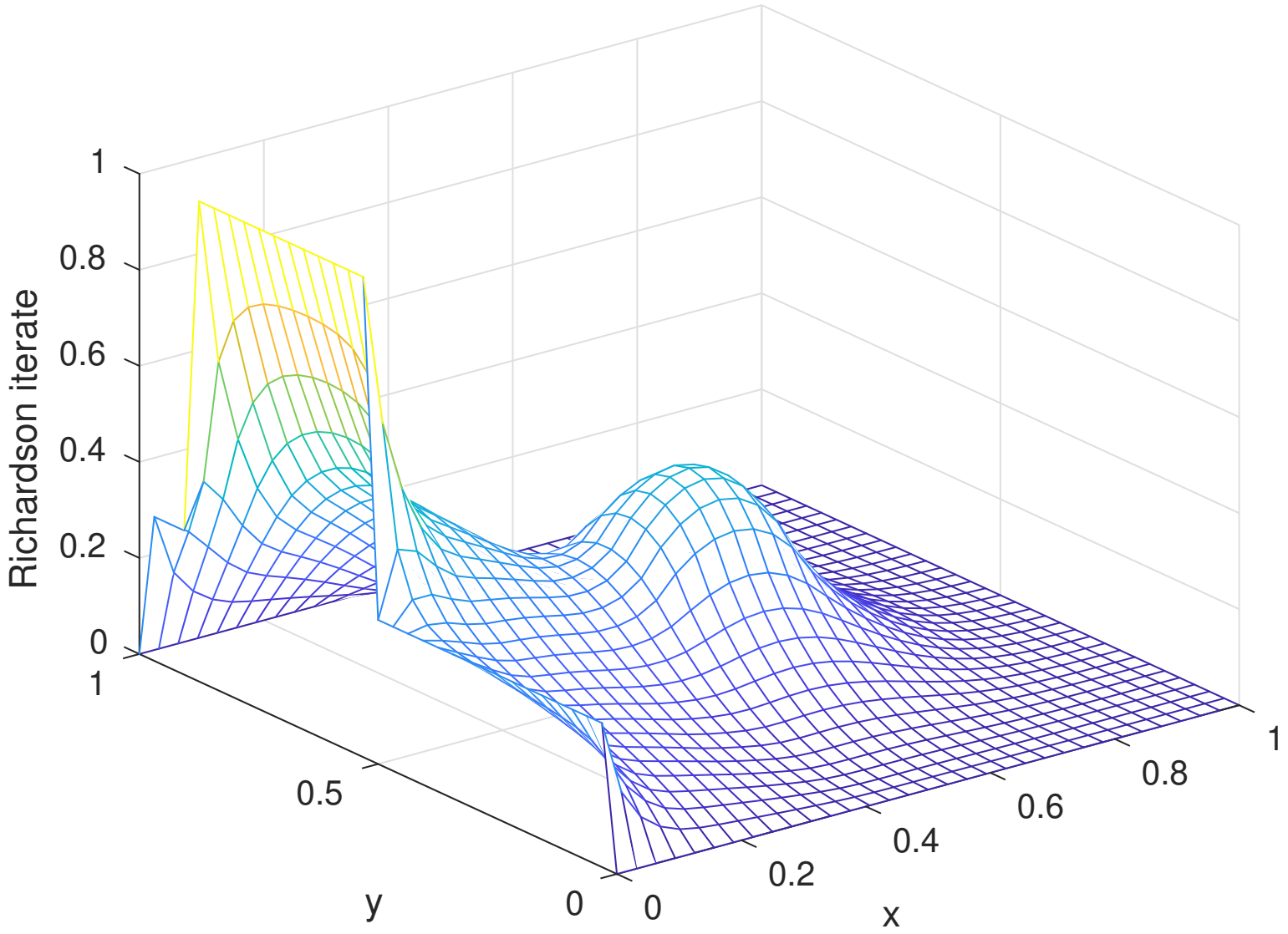
Richardson iteration 18



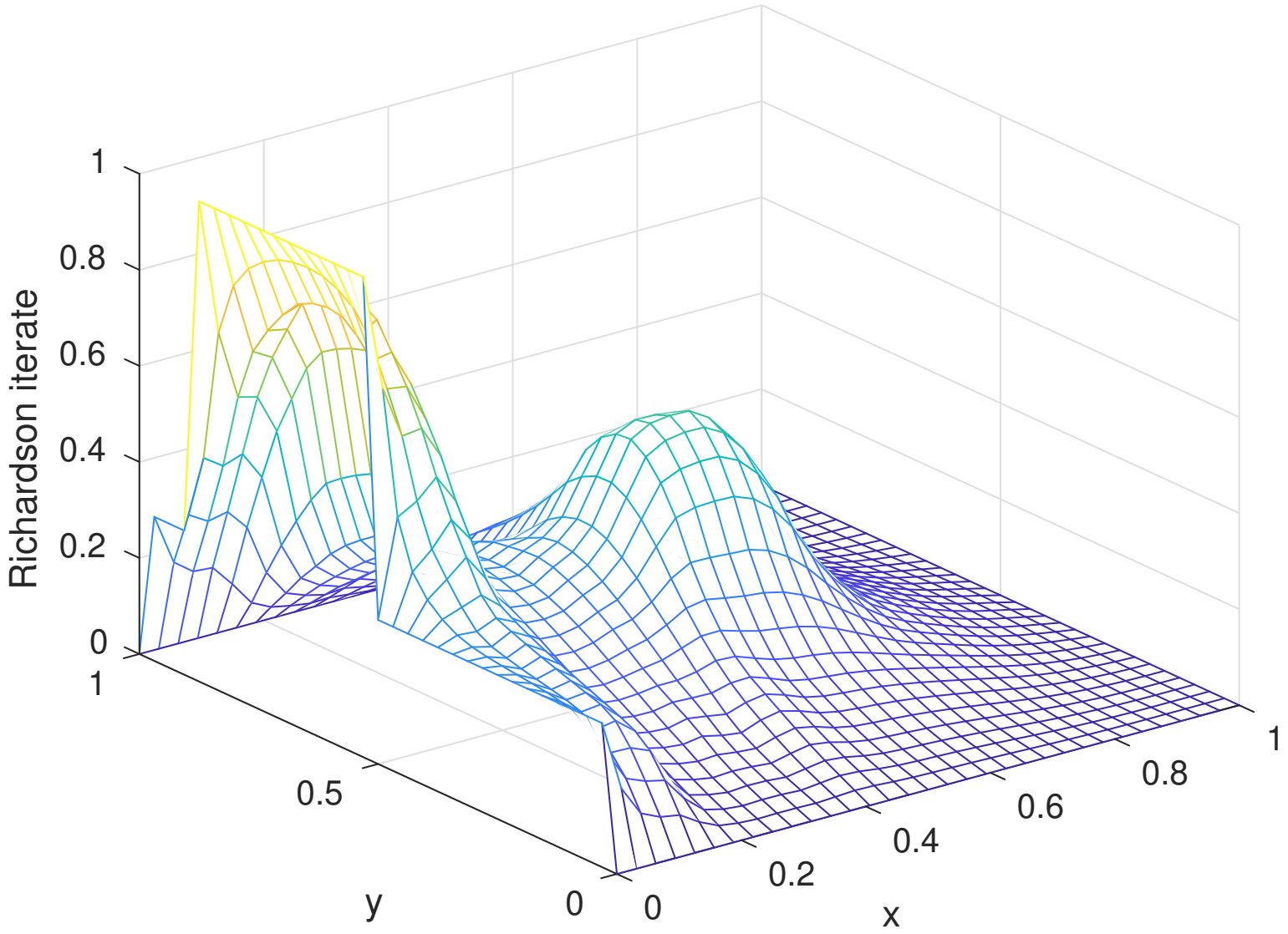
Richardson iteration 19



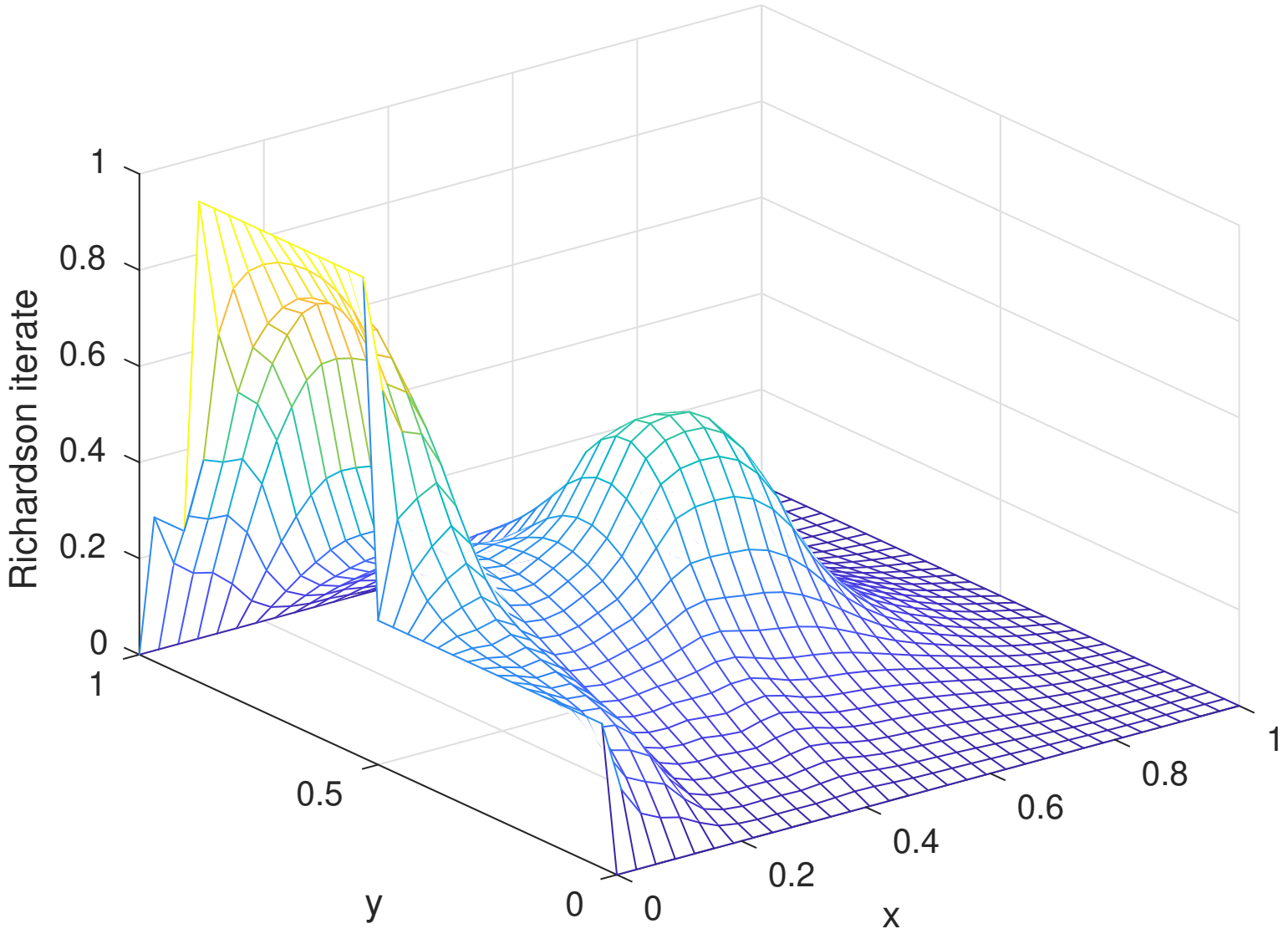
Richardson iteration 20



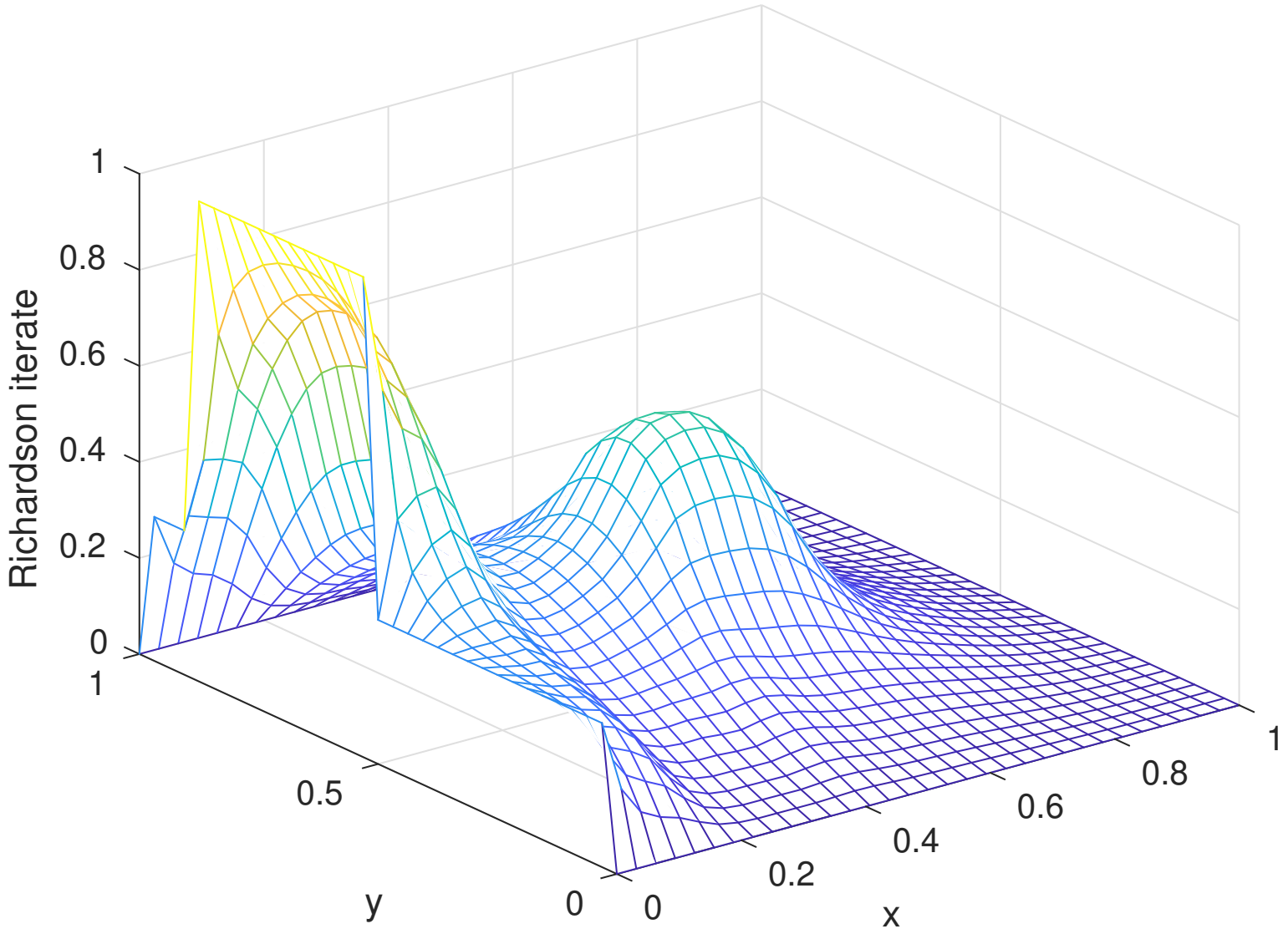
Richardson iteration 21



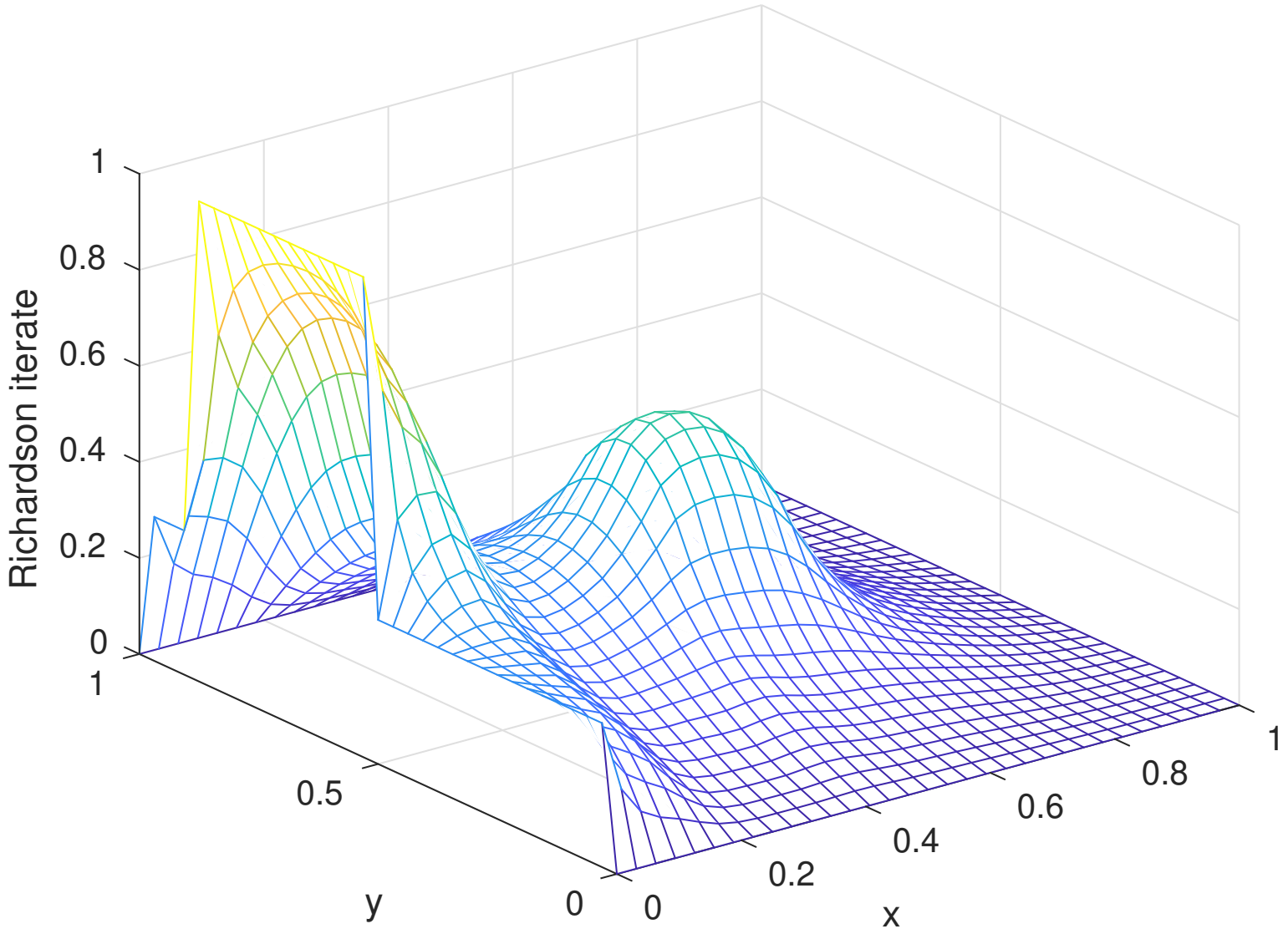
Richardson iteration 22



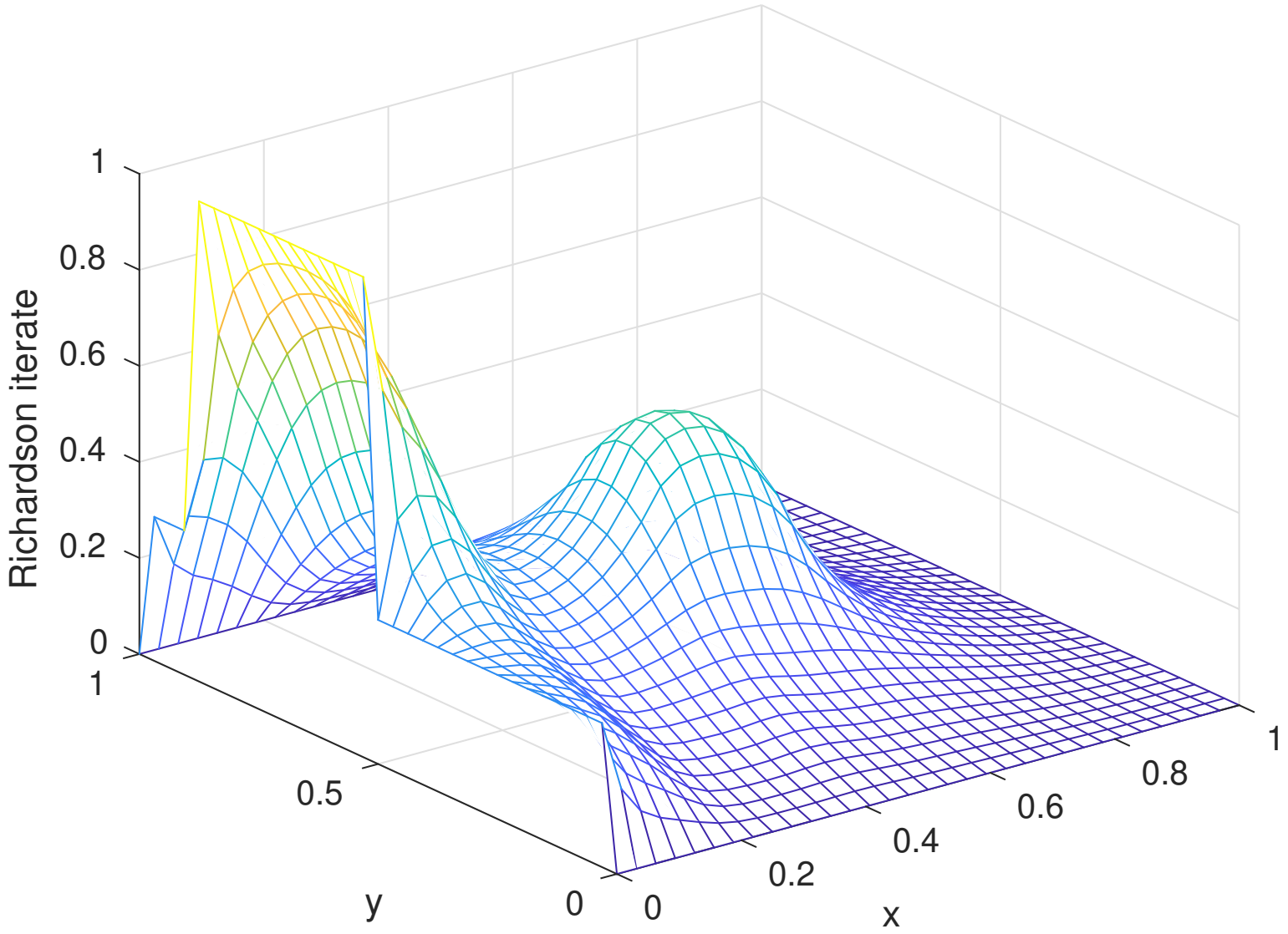
Richardson iteration 23



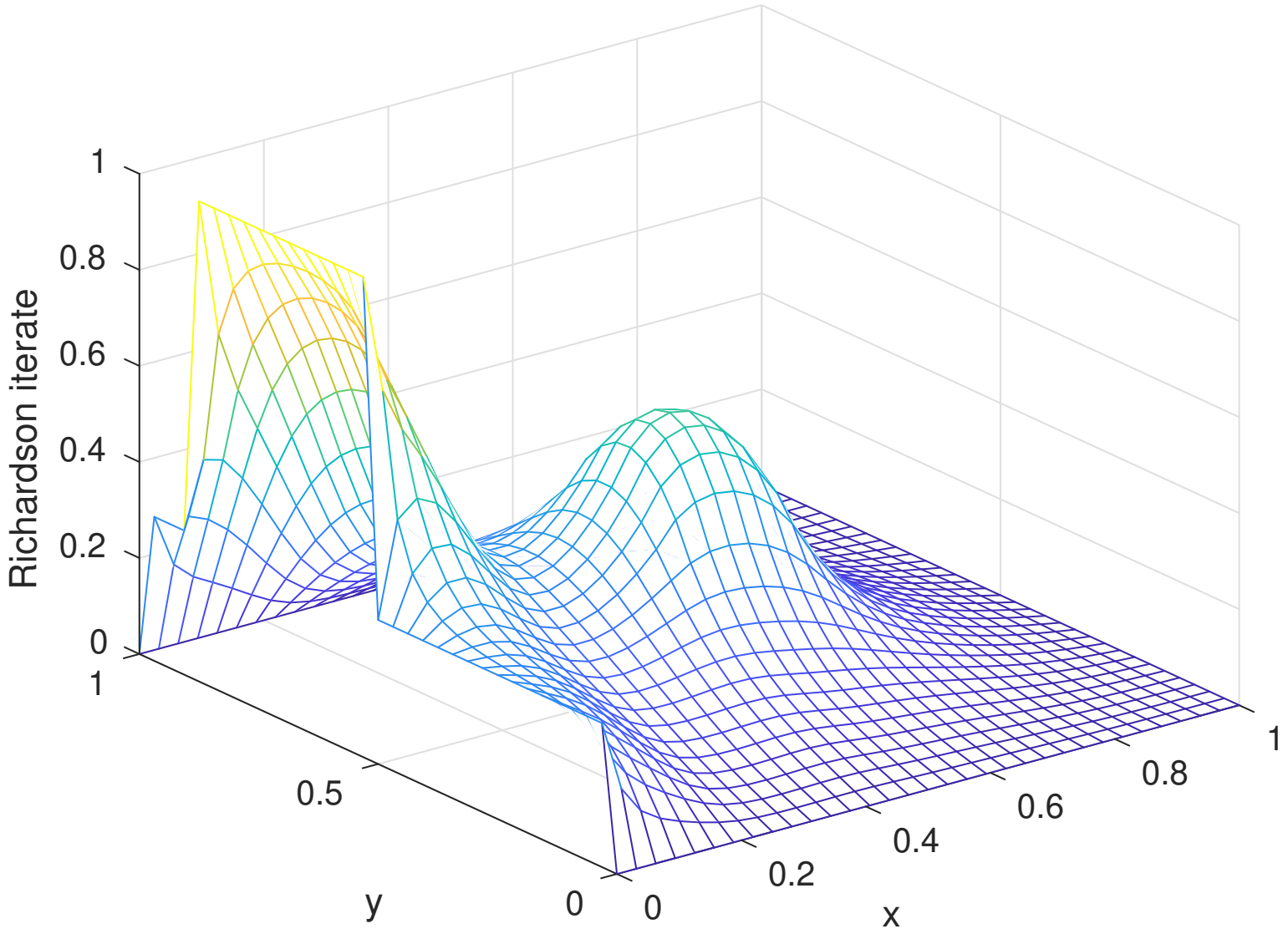
Richardson iteration 24



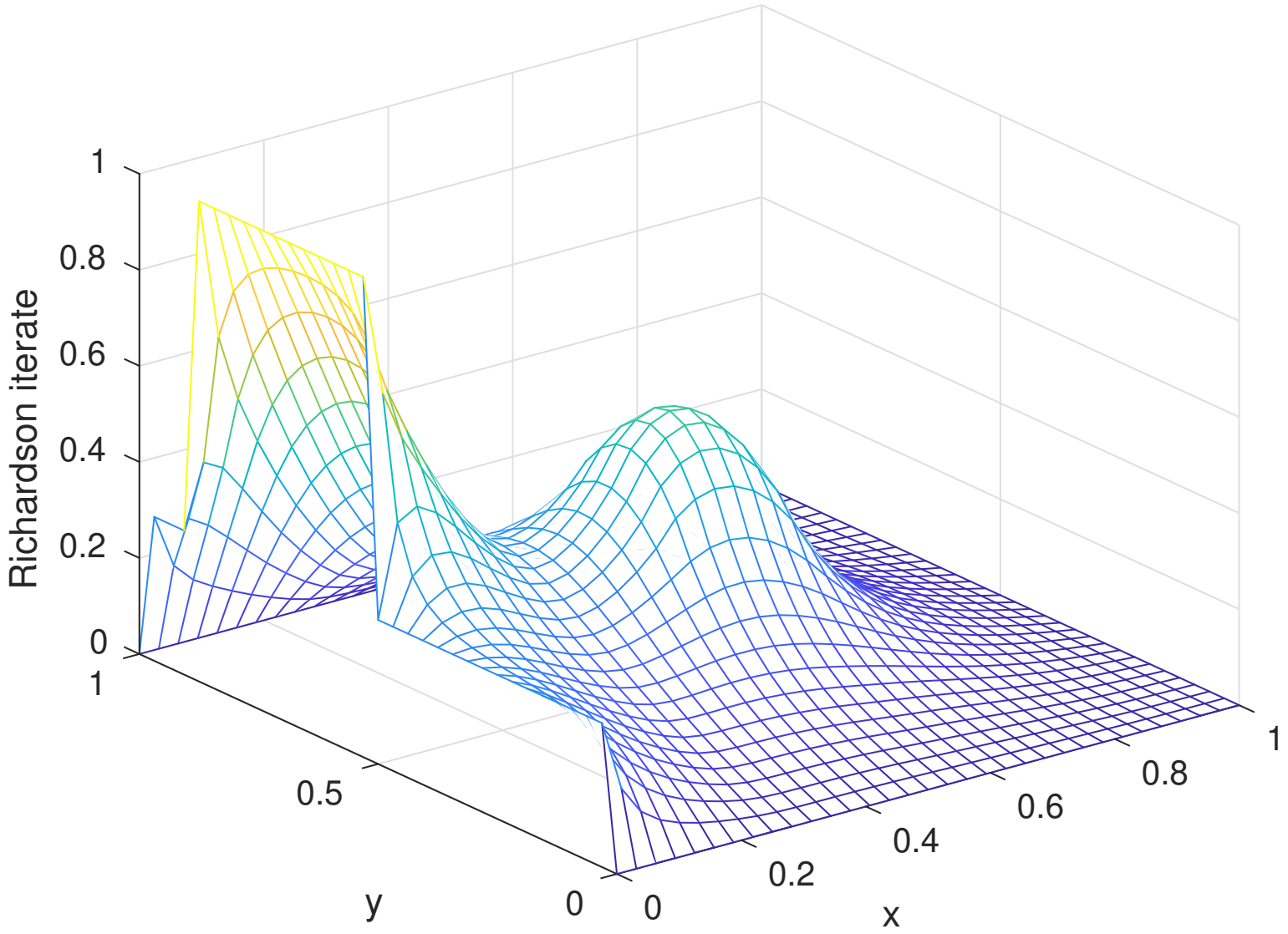
Richardson iteration 25



Richardson iteration 26



Richardson iteration 27



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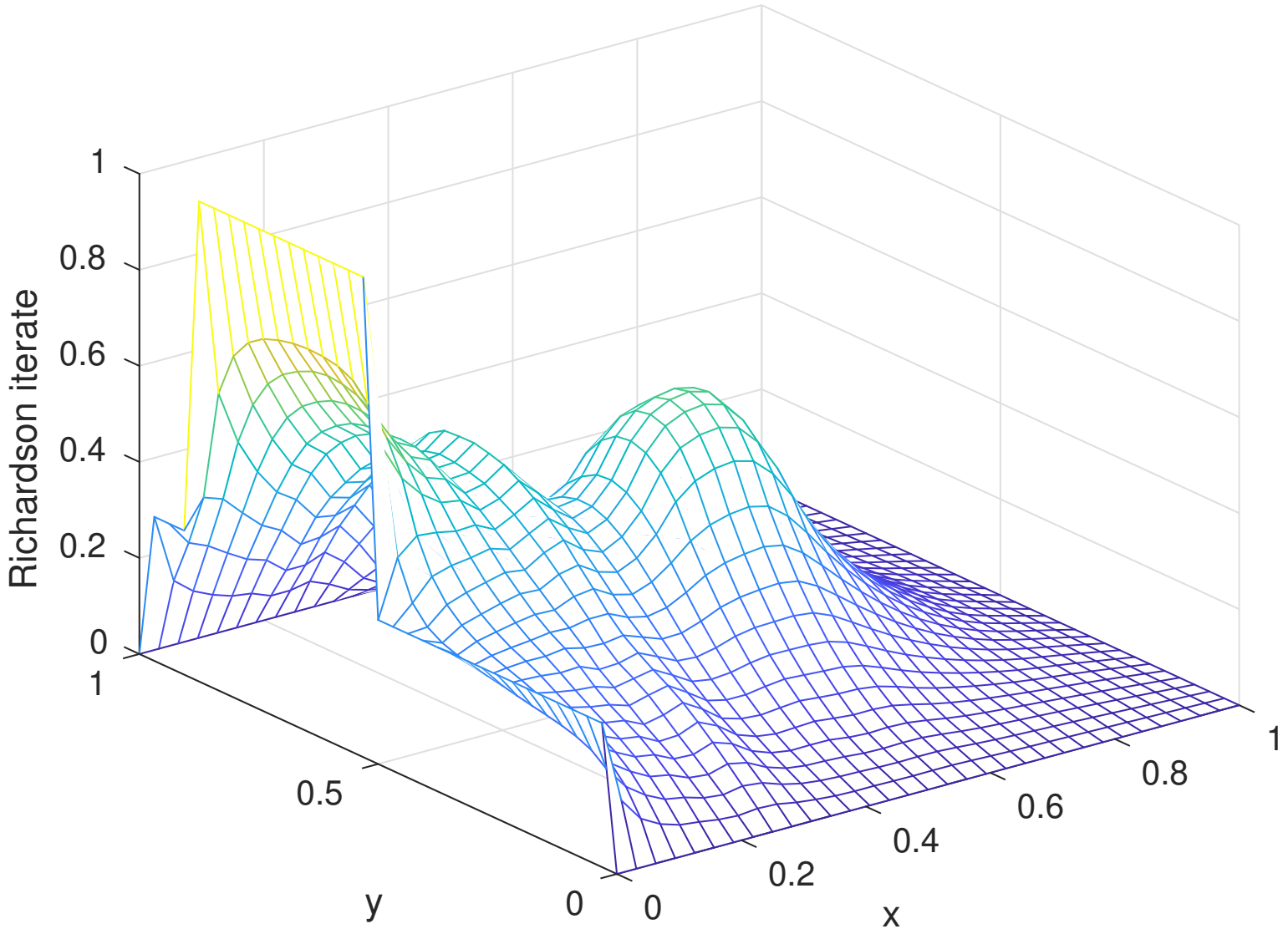

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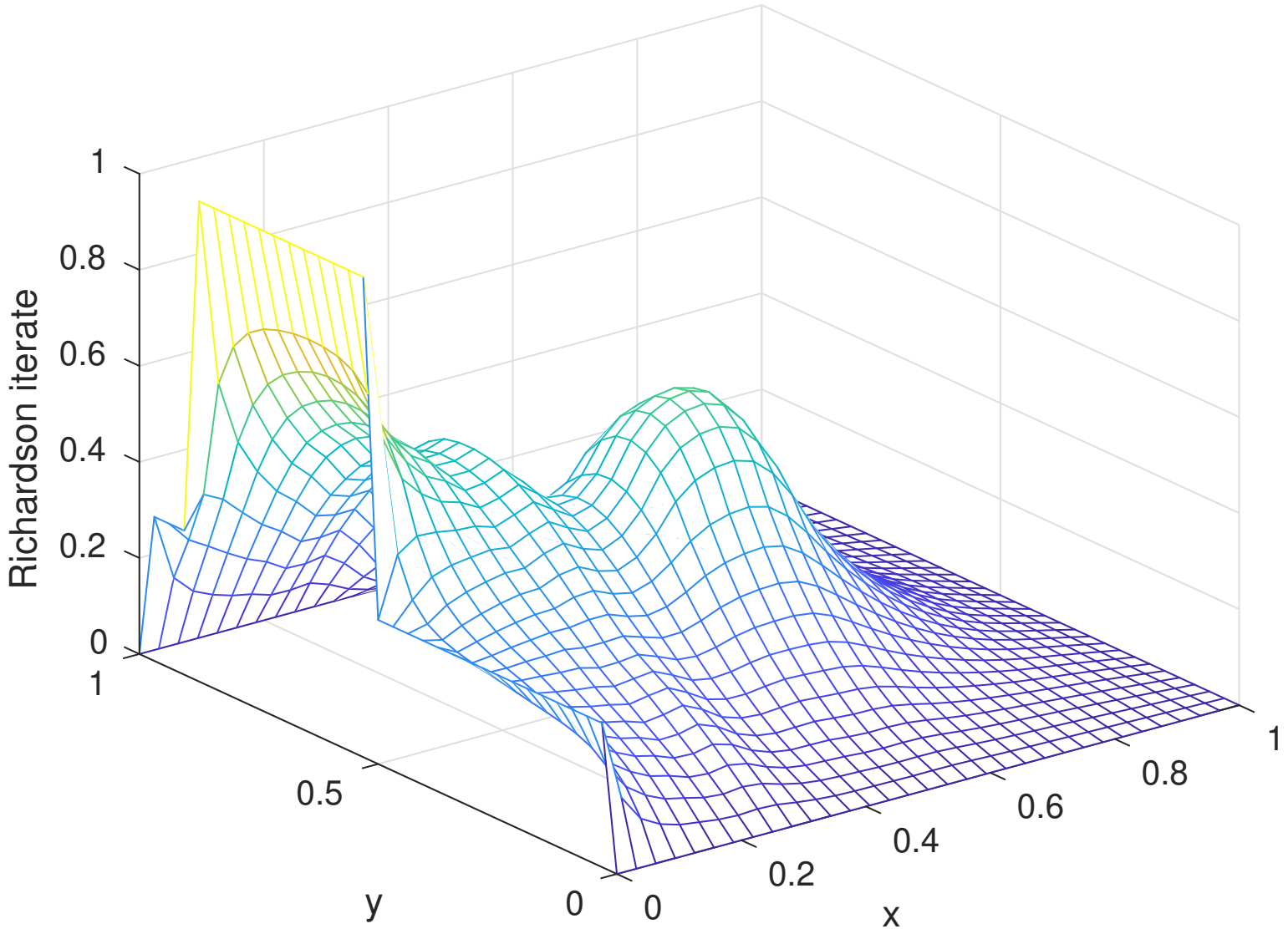

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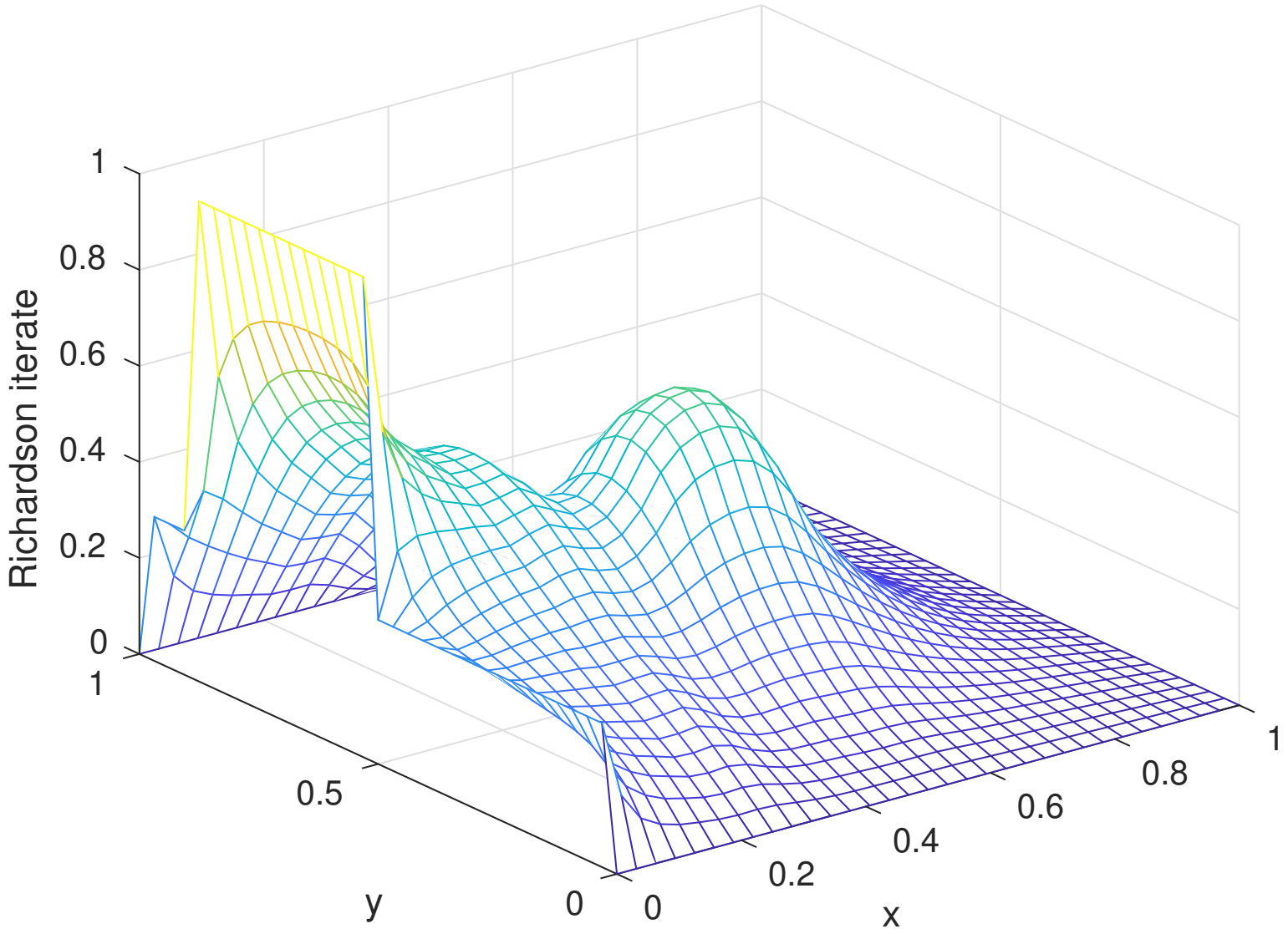
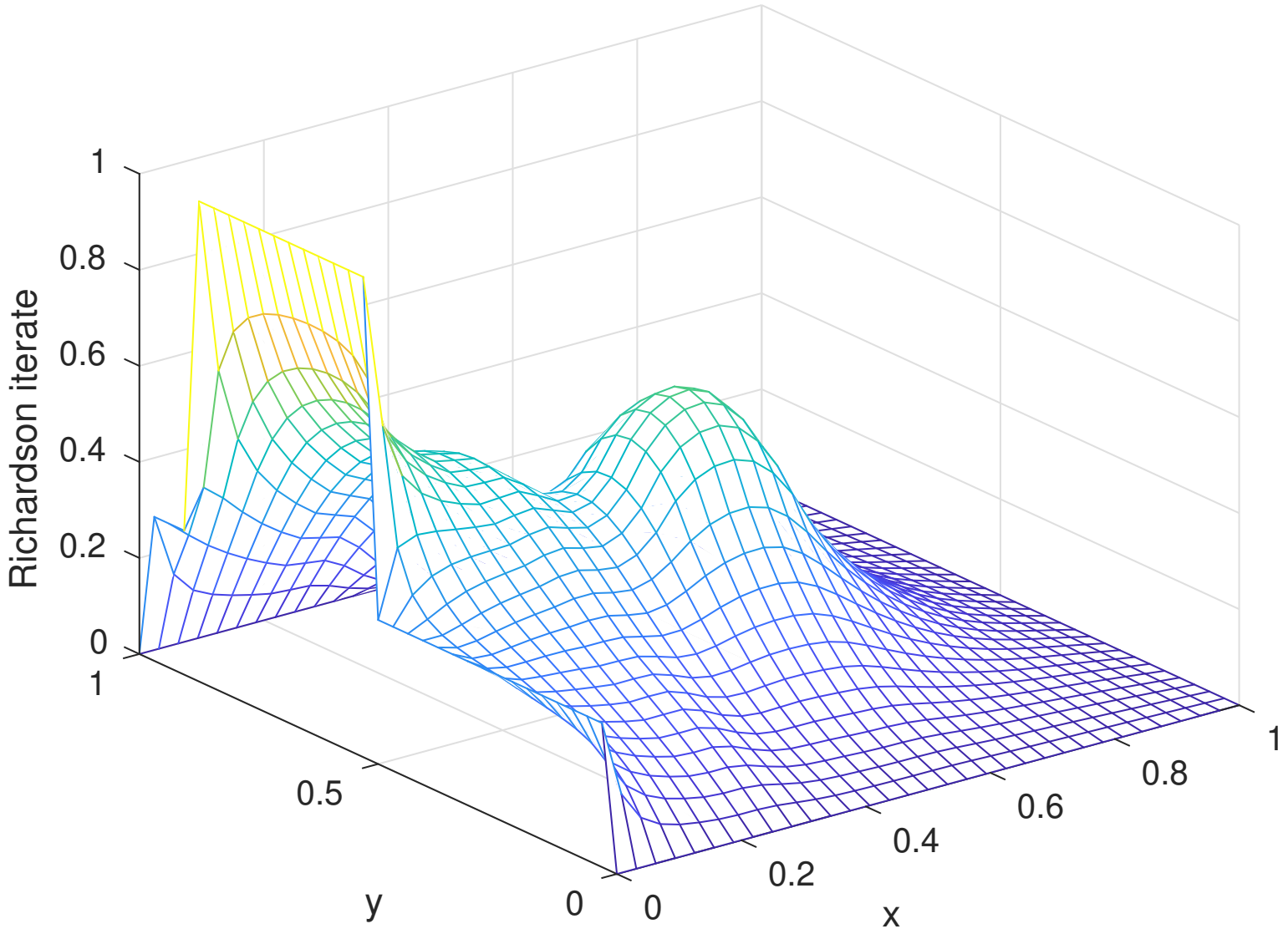
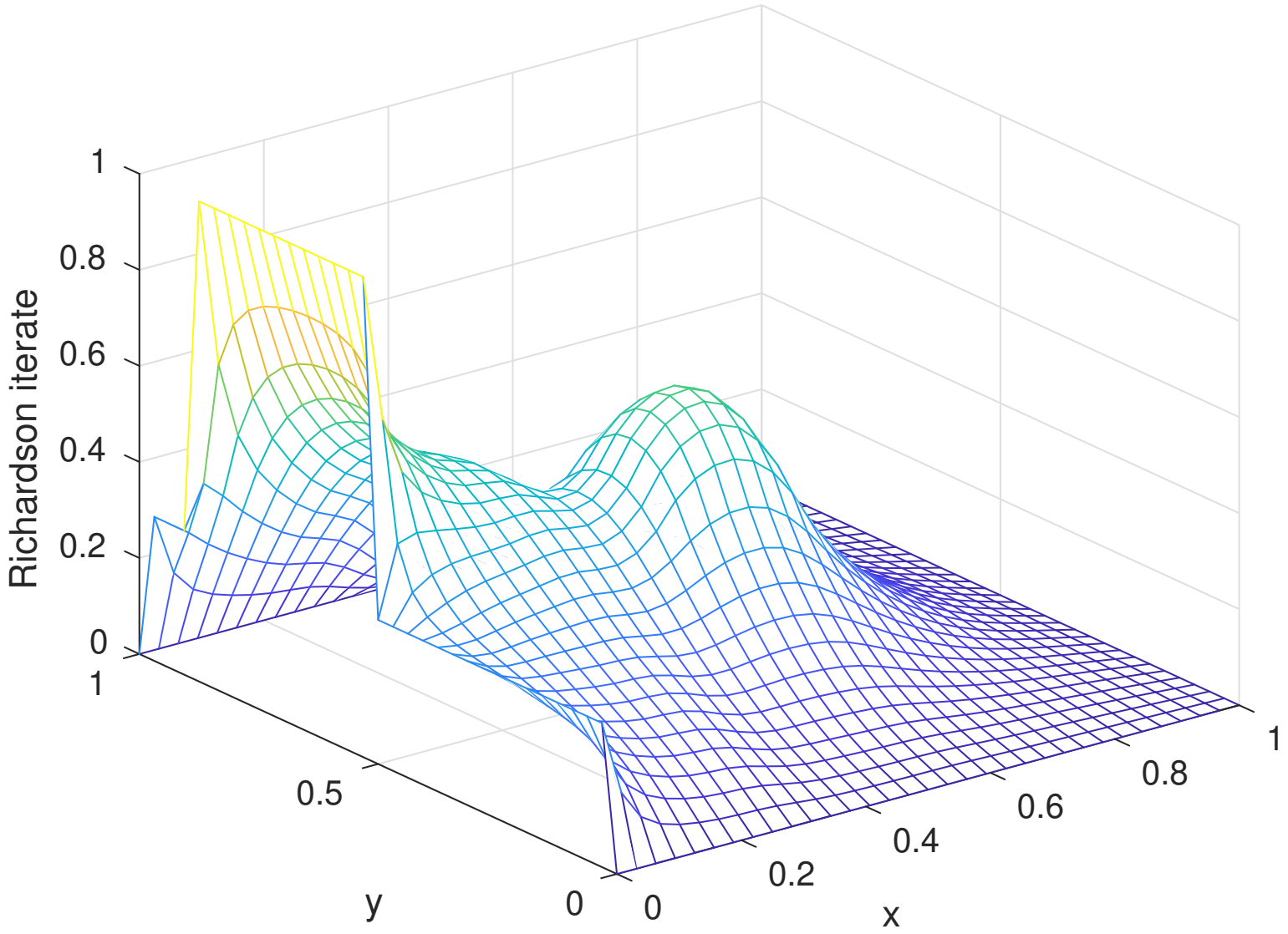


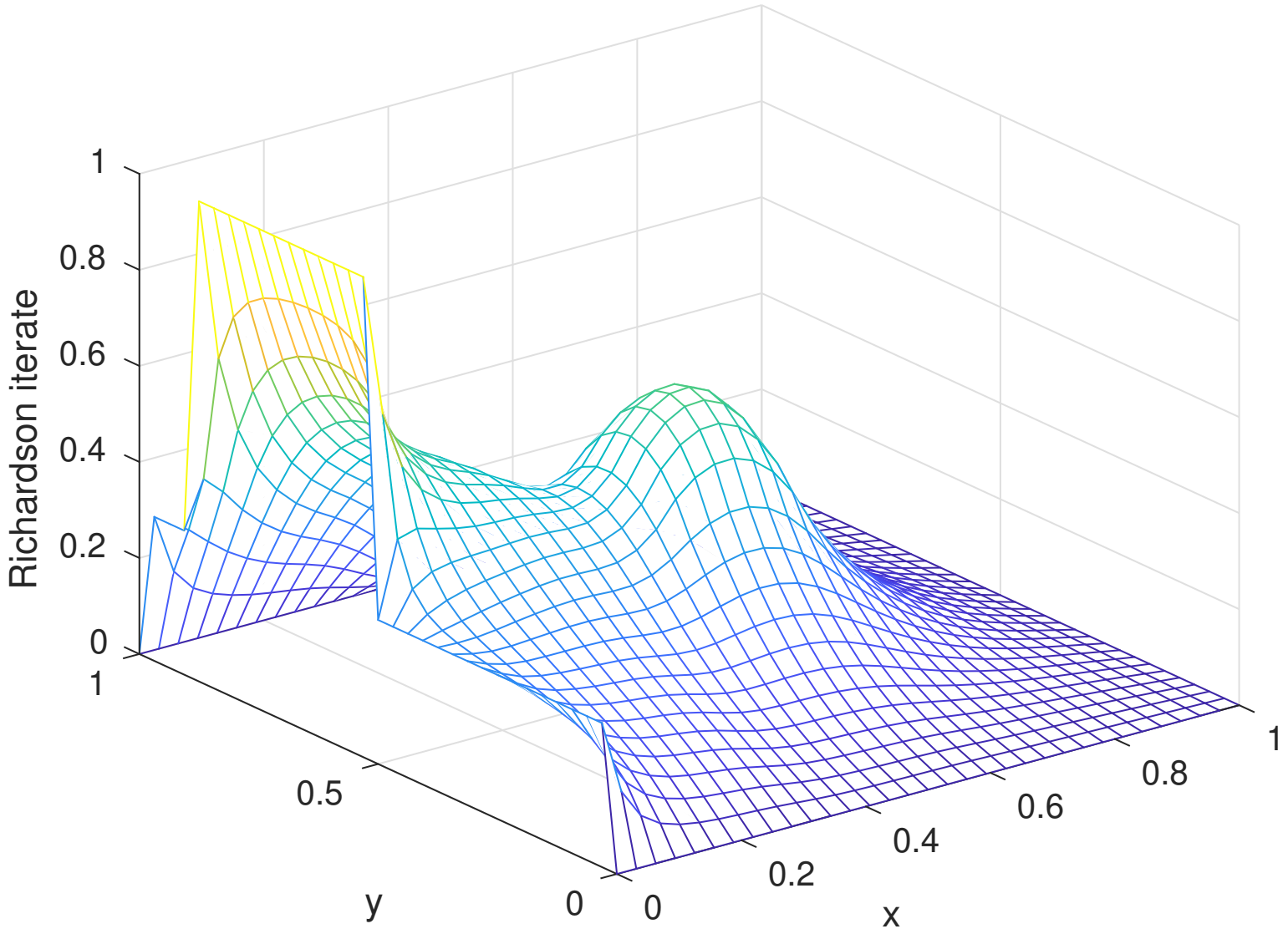
Richardson iteration 31



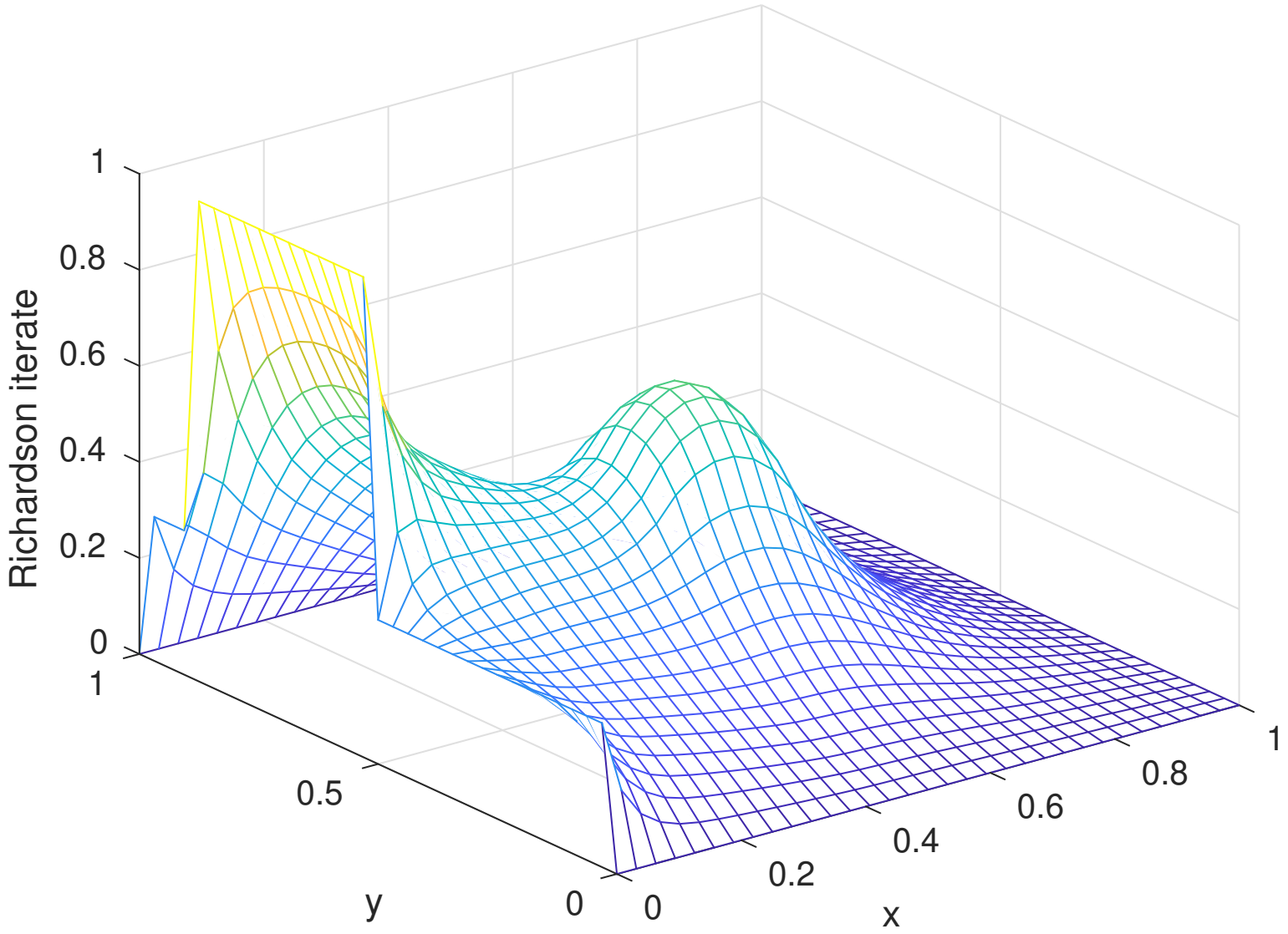
Richardson iteration 32



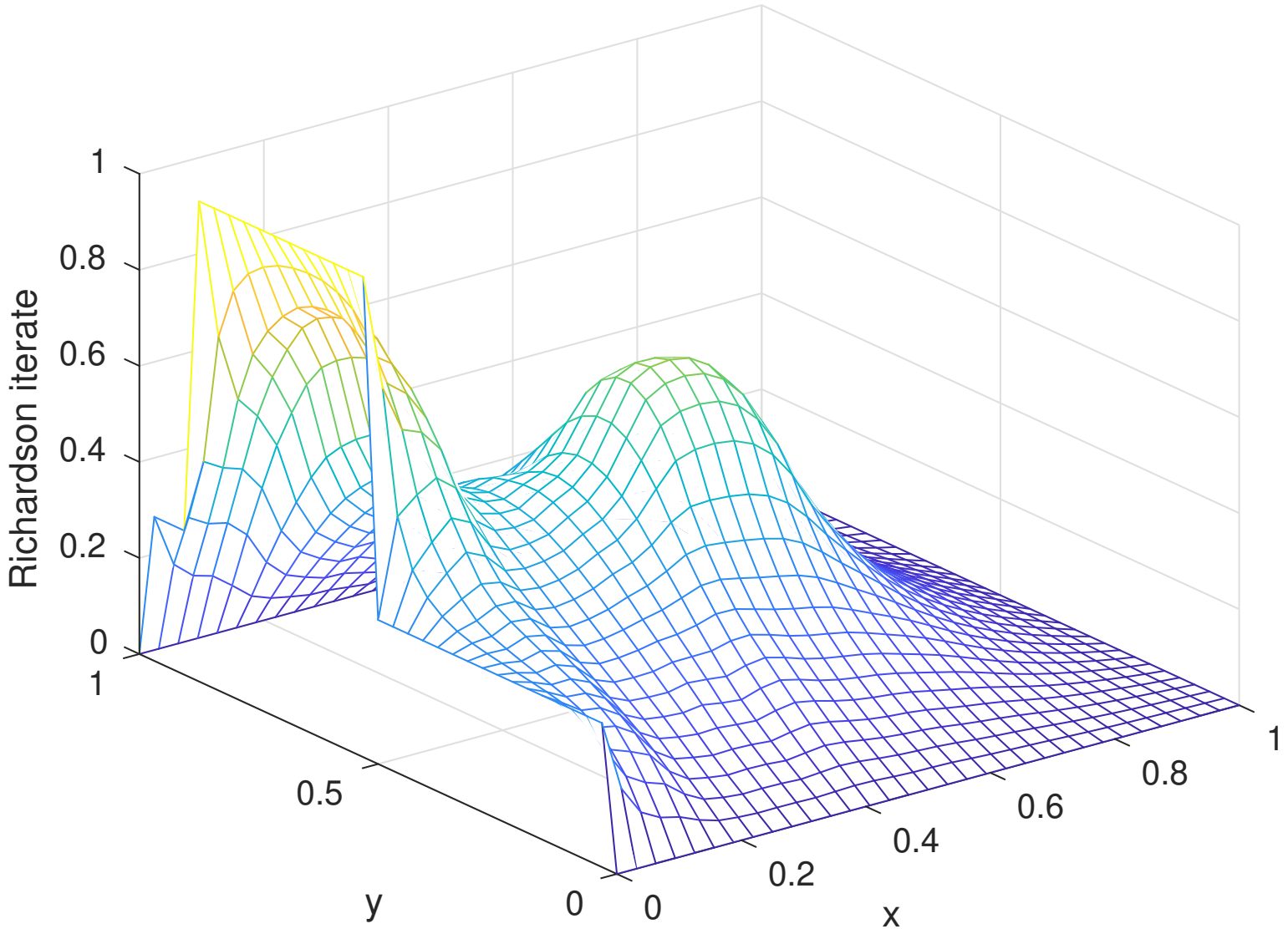
Richardson iteration 33



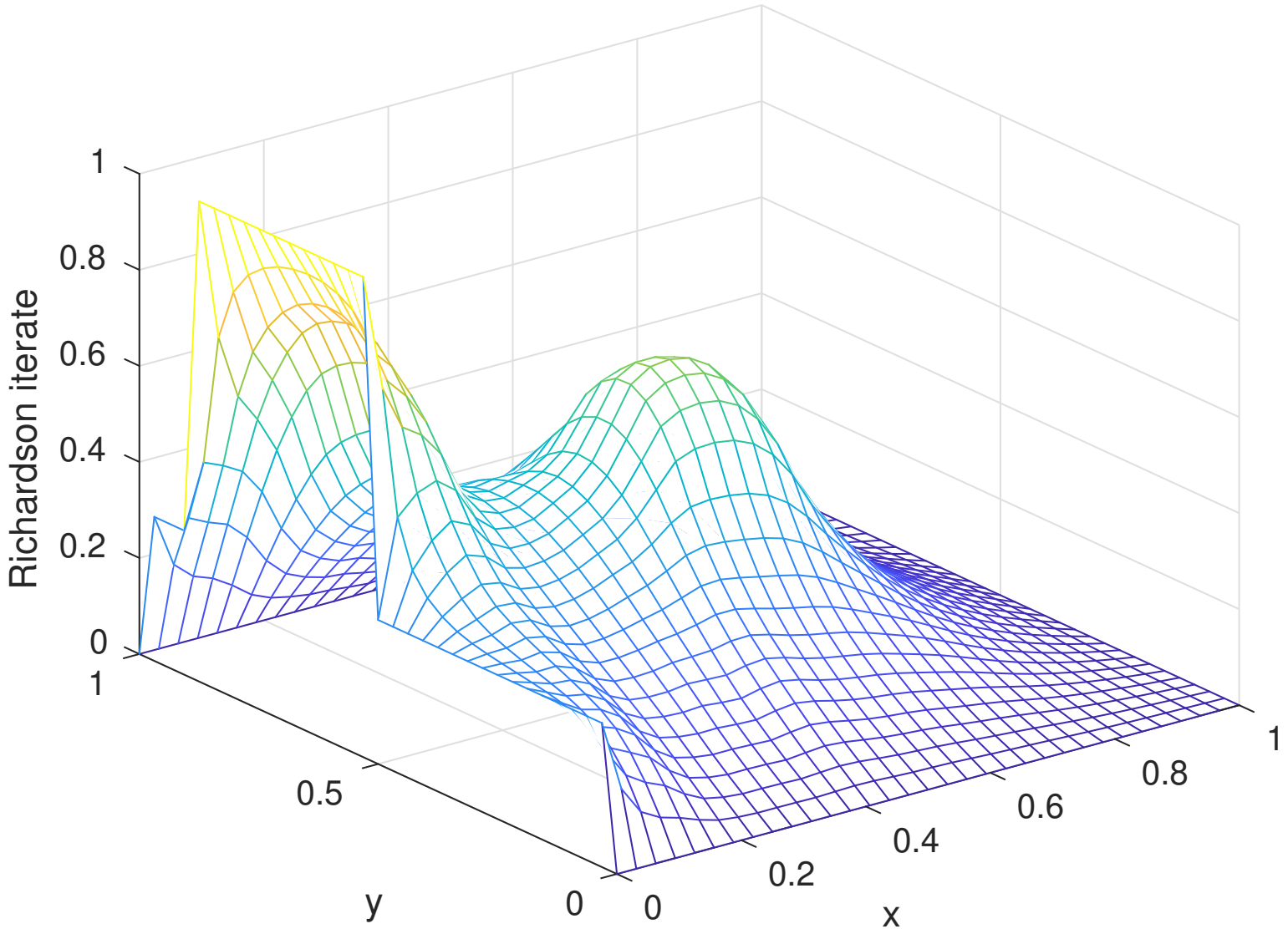
Richardson iteration 34



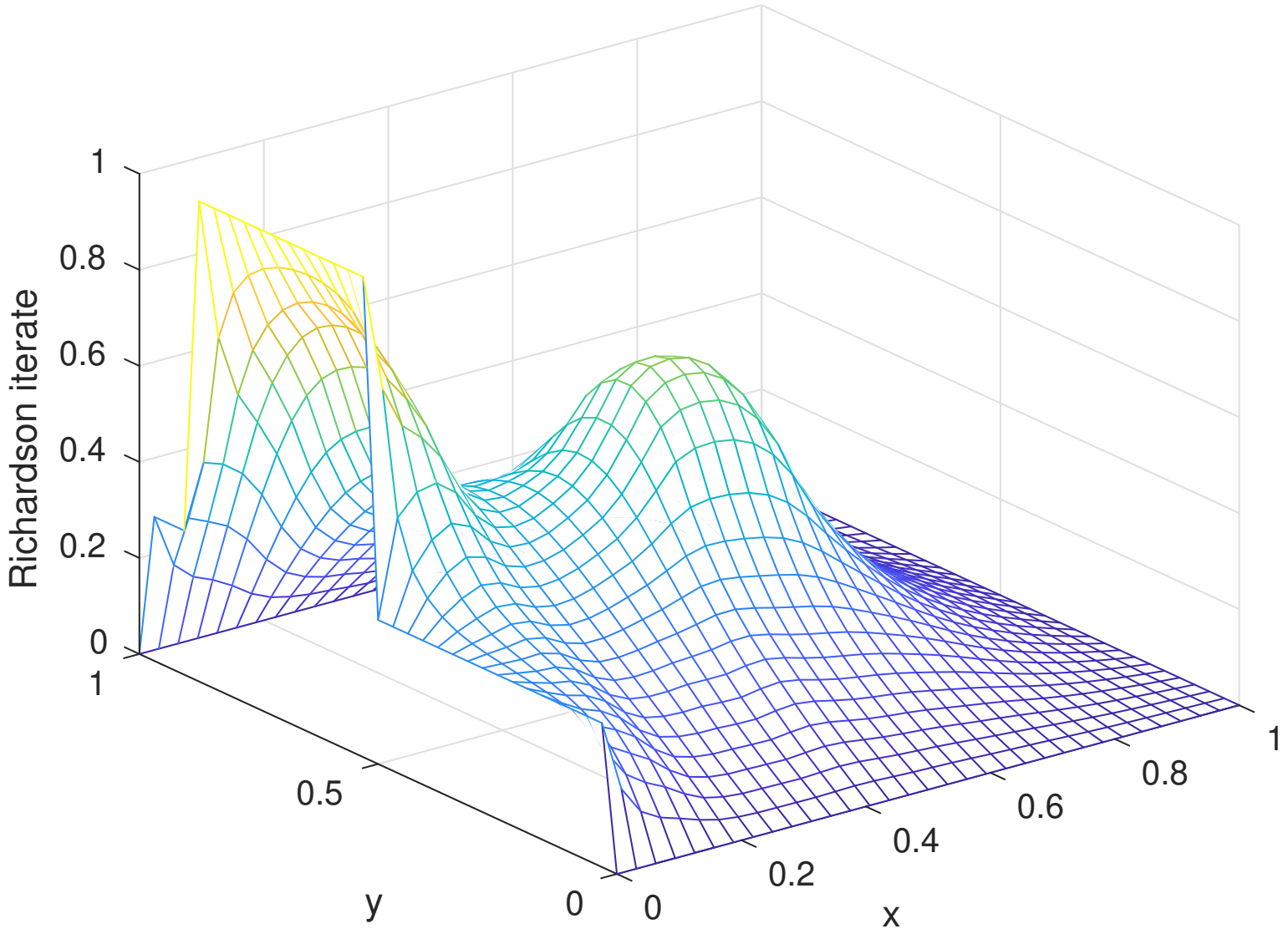
Richardson iteration 35



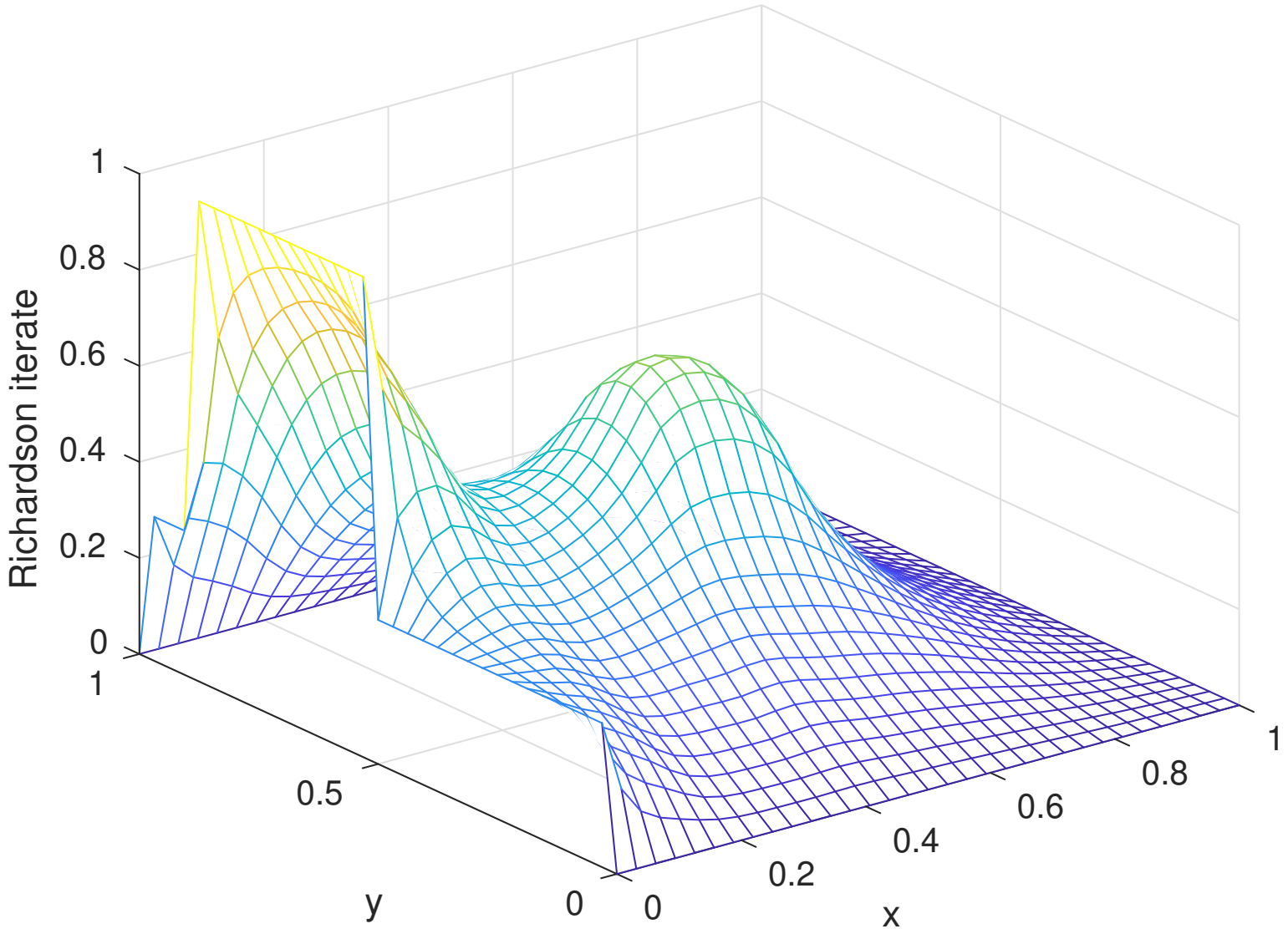
Richardson iteration 36



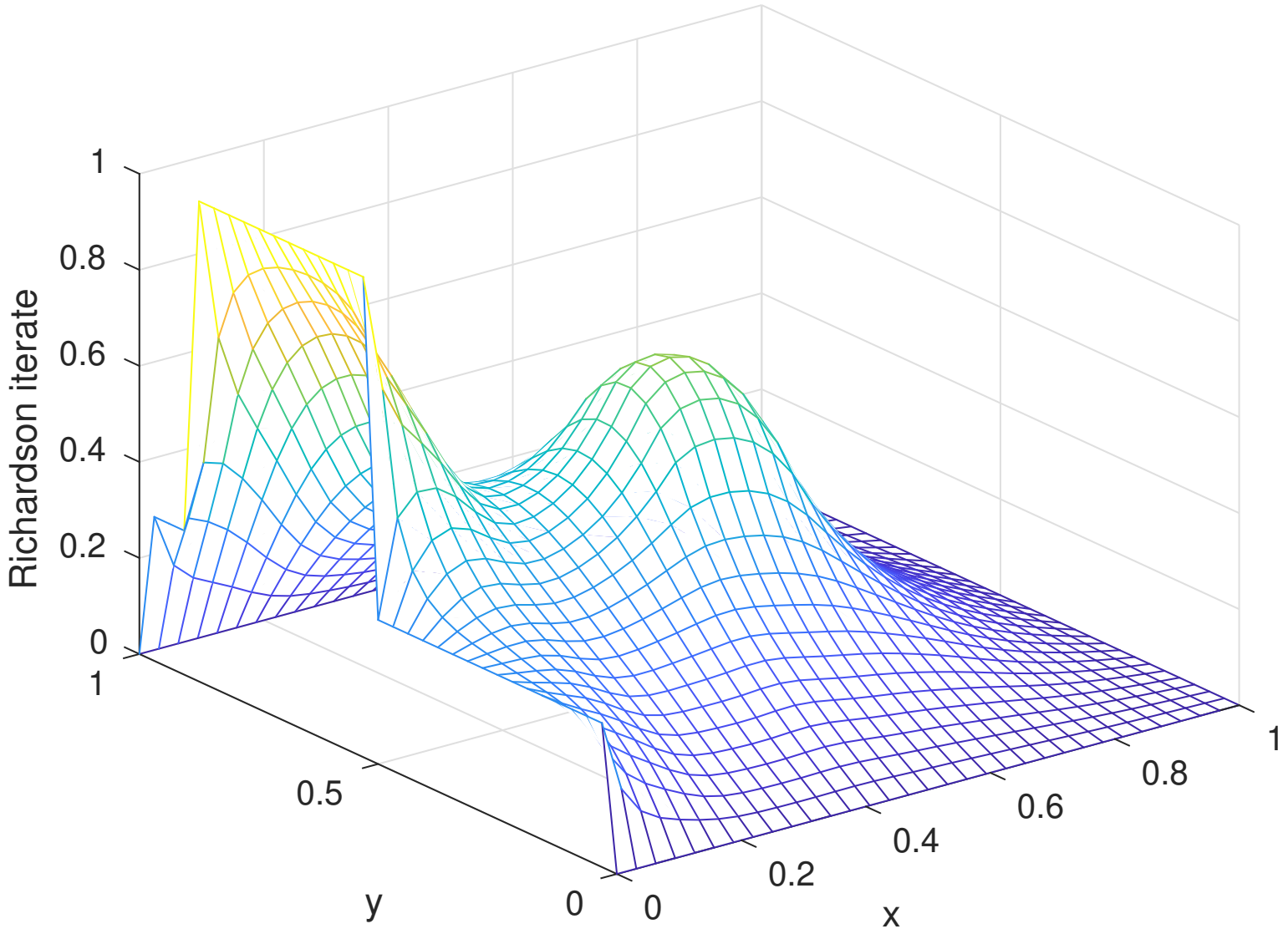
Richardson iteration 37



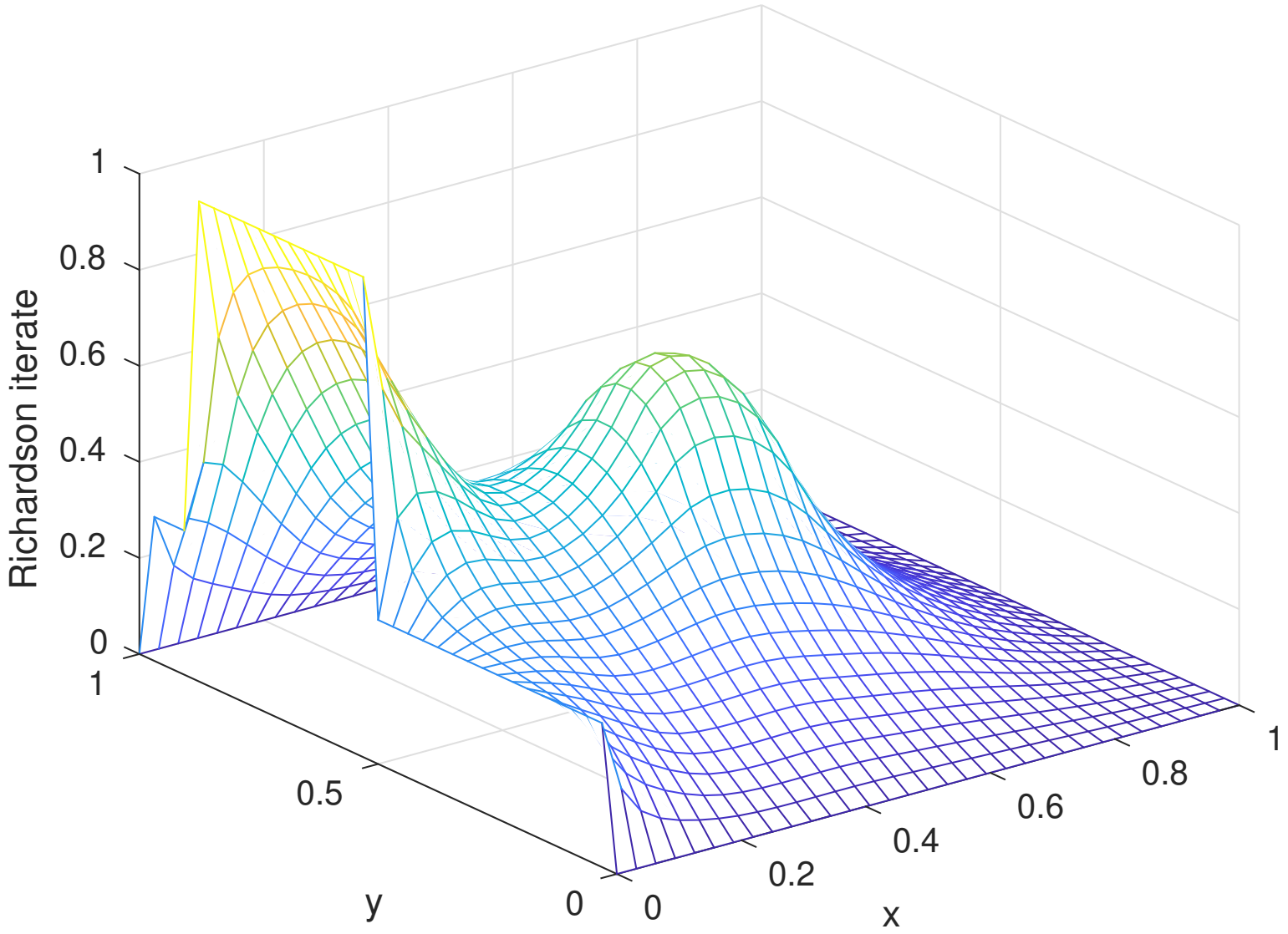
Richardson iteration 38



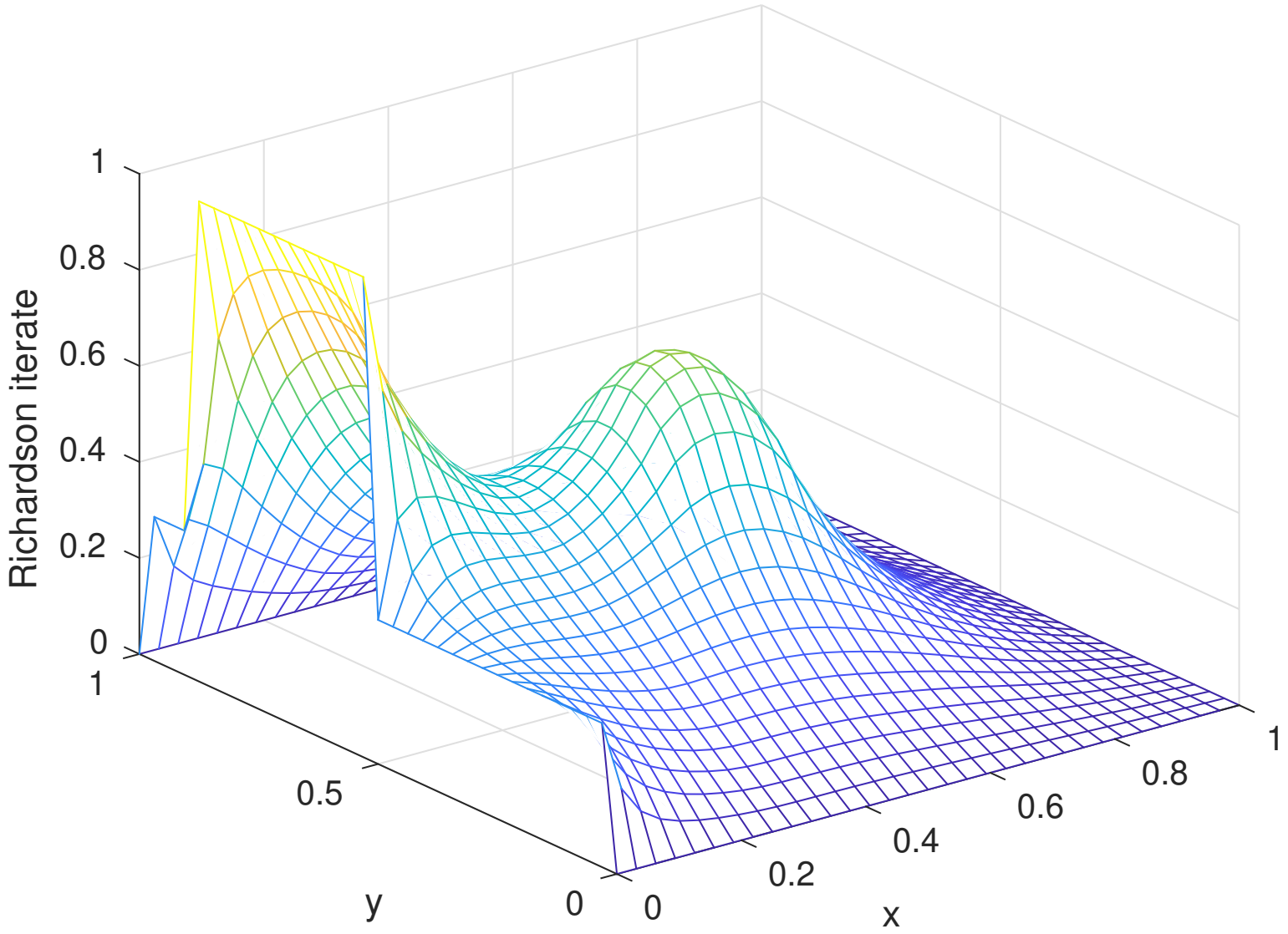
Richardson iteration 39



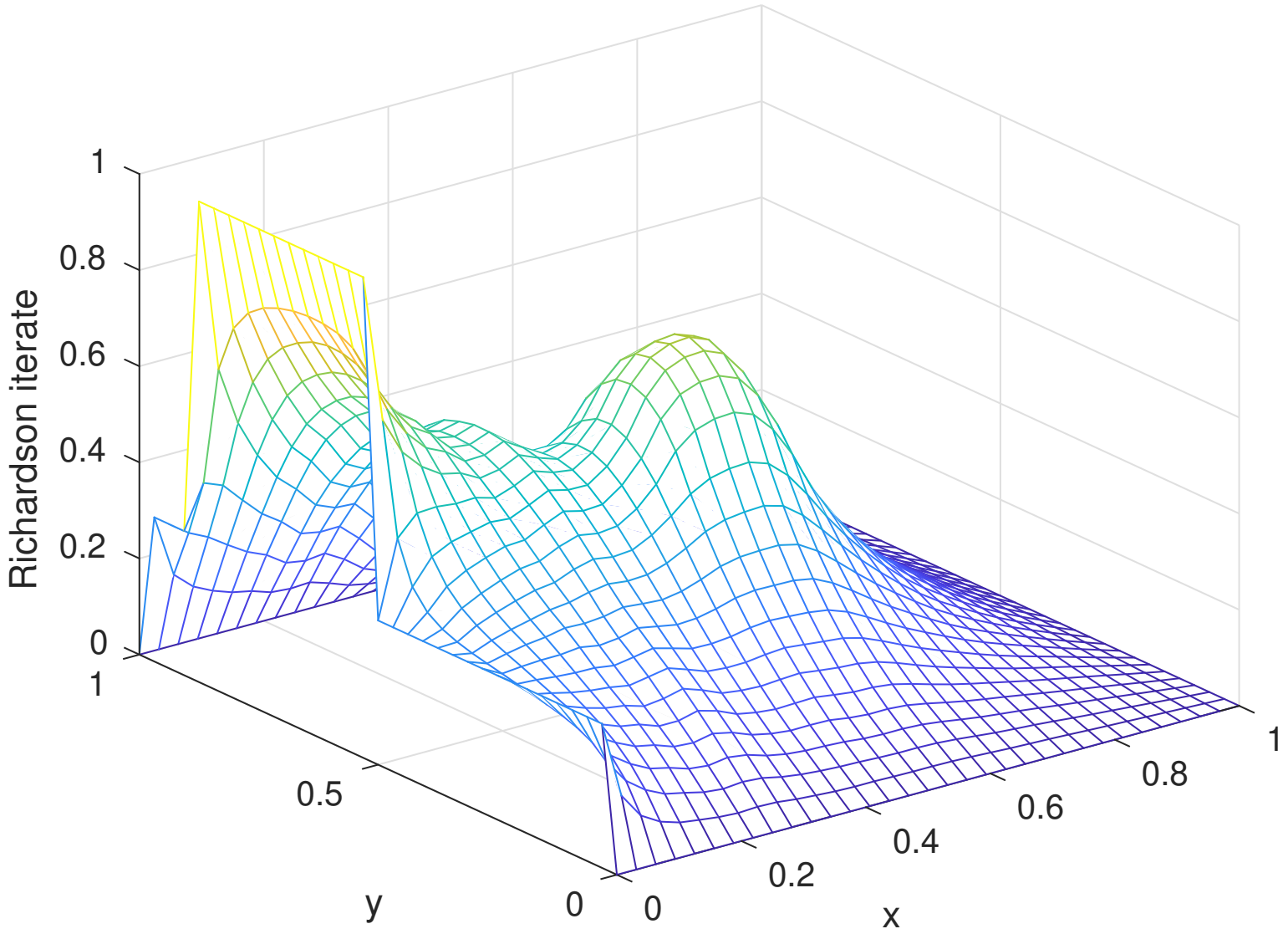
Richardson iteration 40



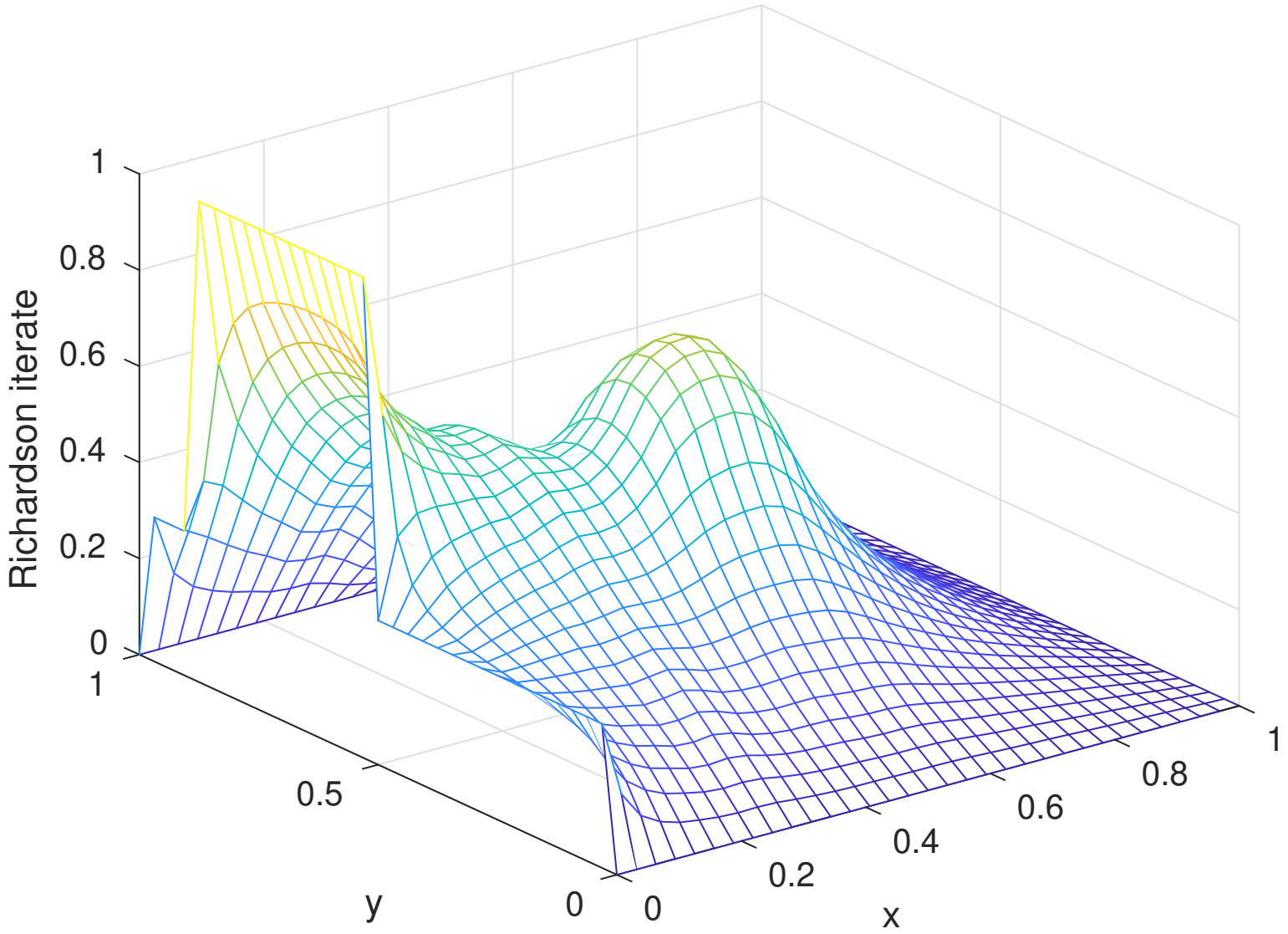
Richardson iteration 41



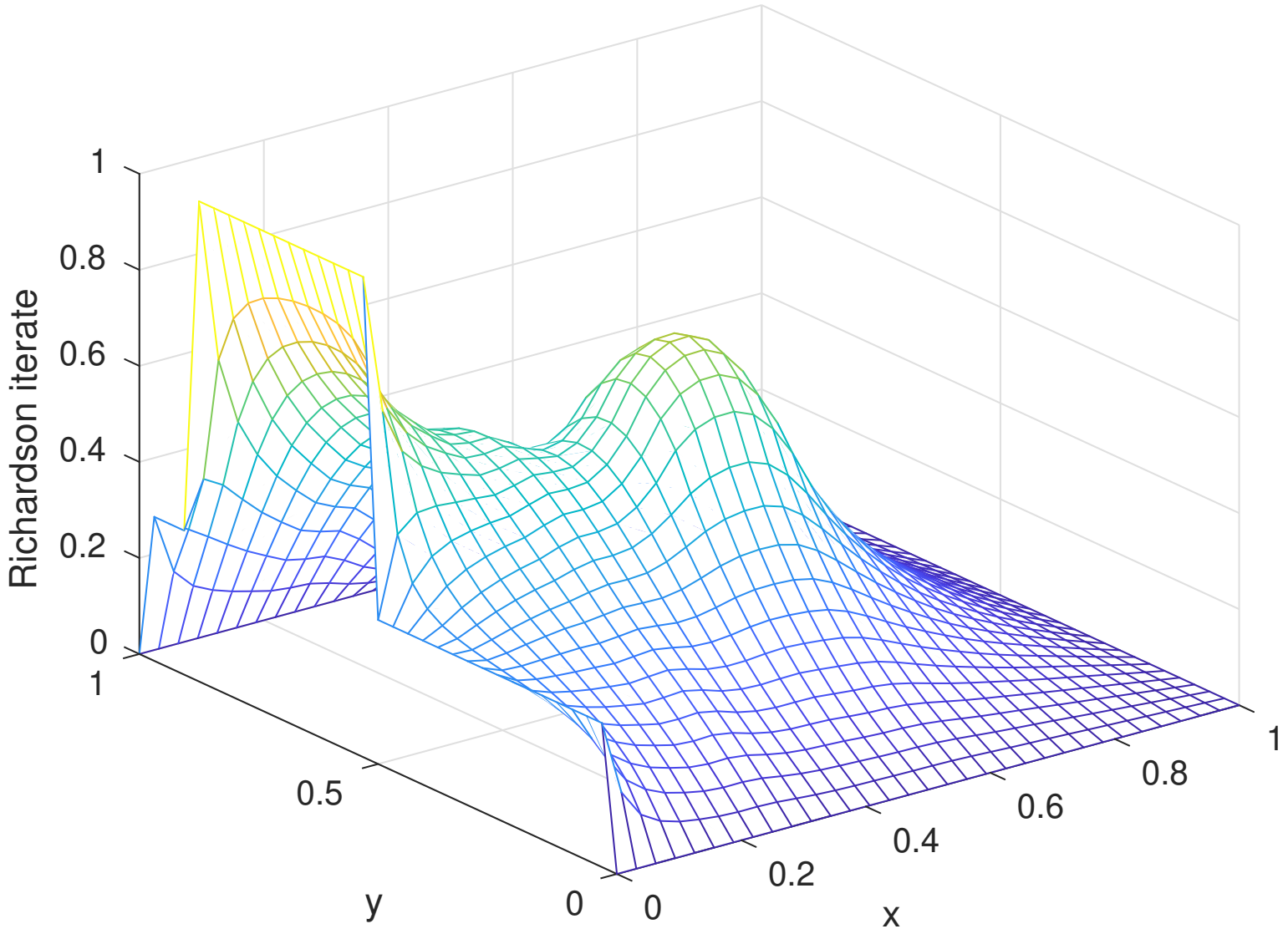
Richardson iteration 42



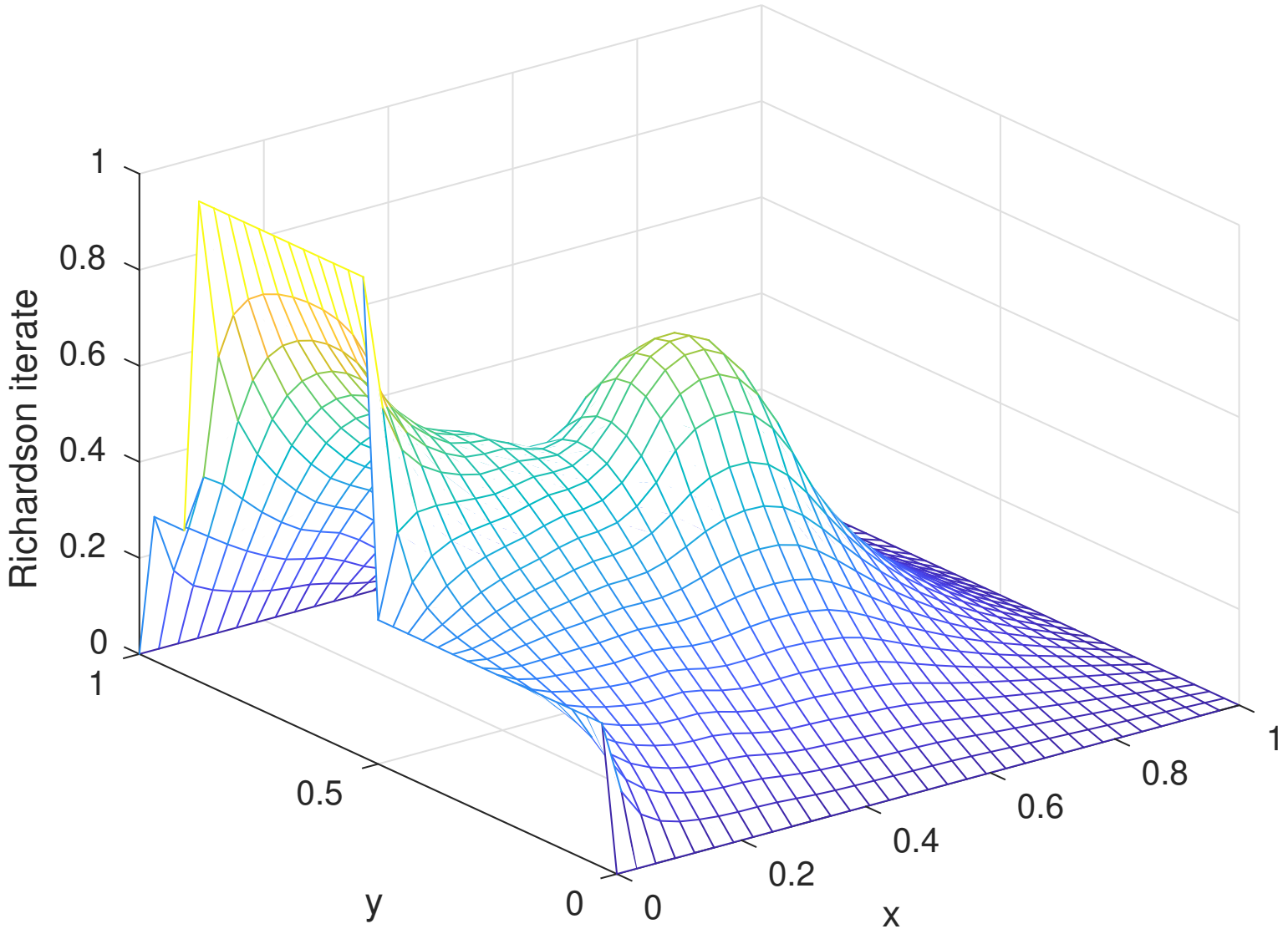
Richardson iteration 43



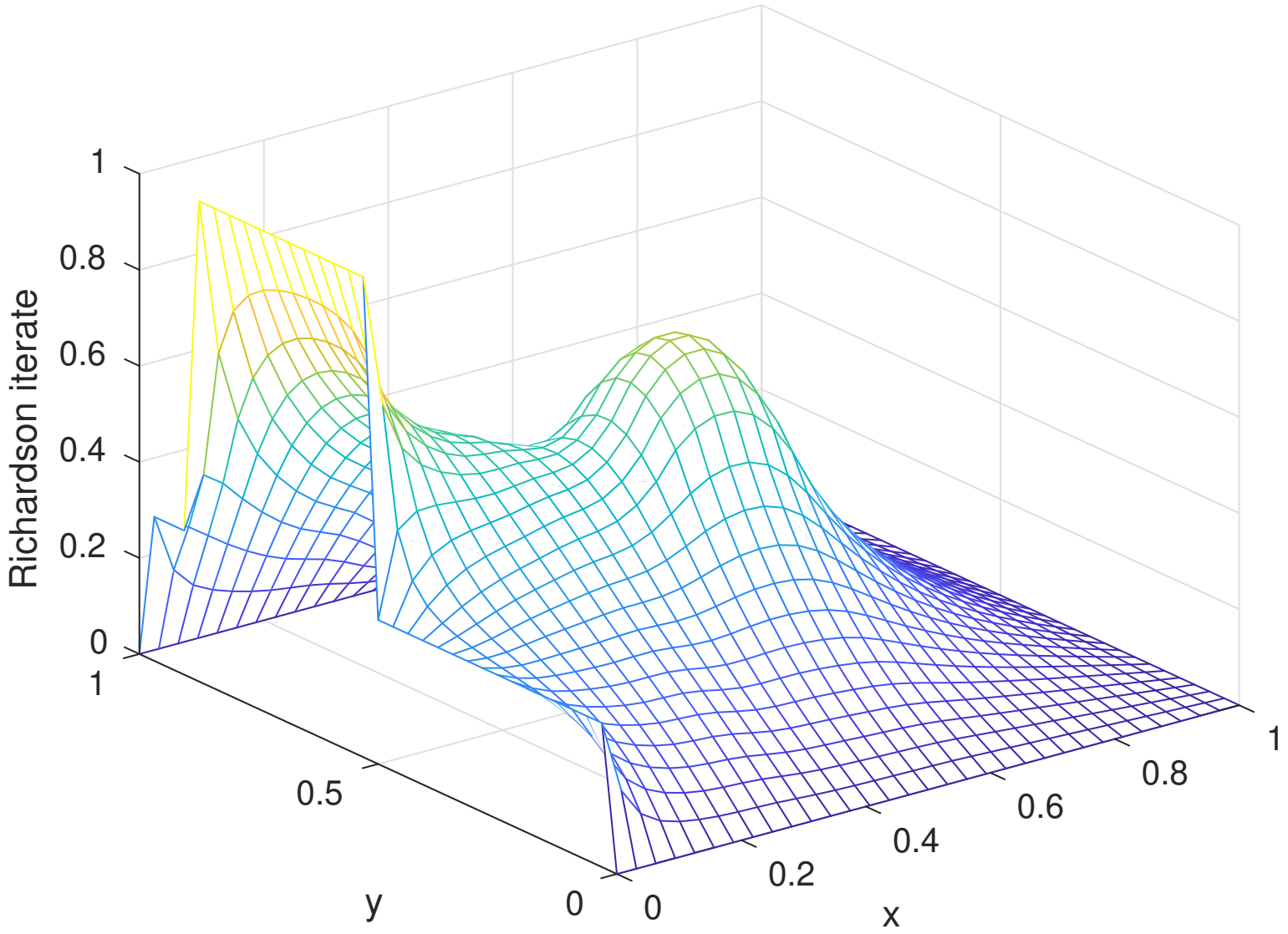
Richardson iteration 44



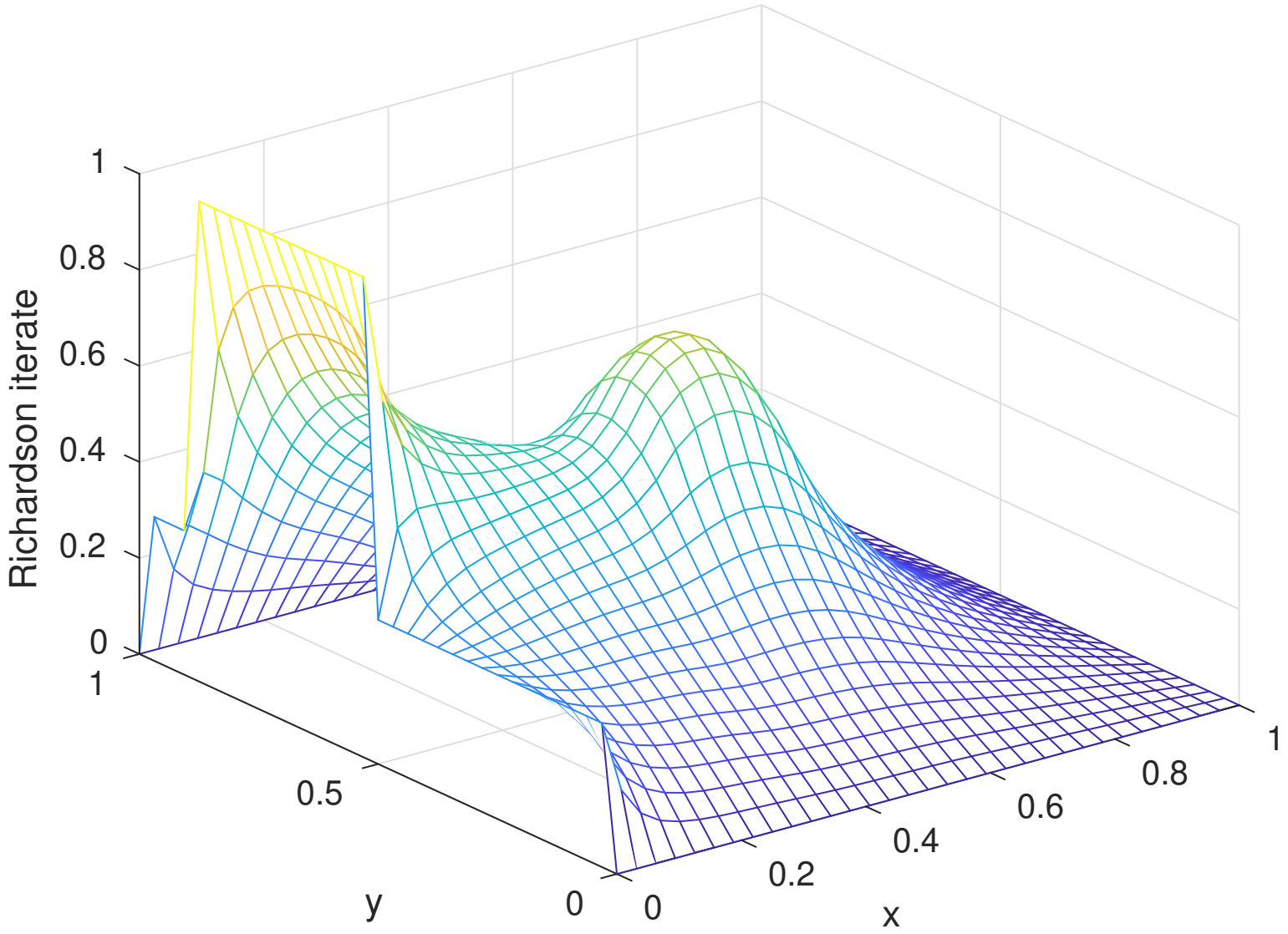
Richardson iteration 45



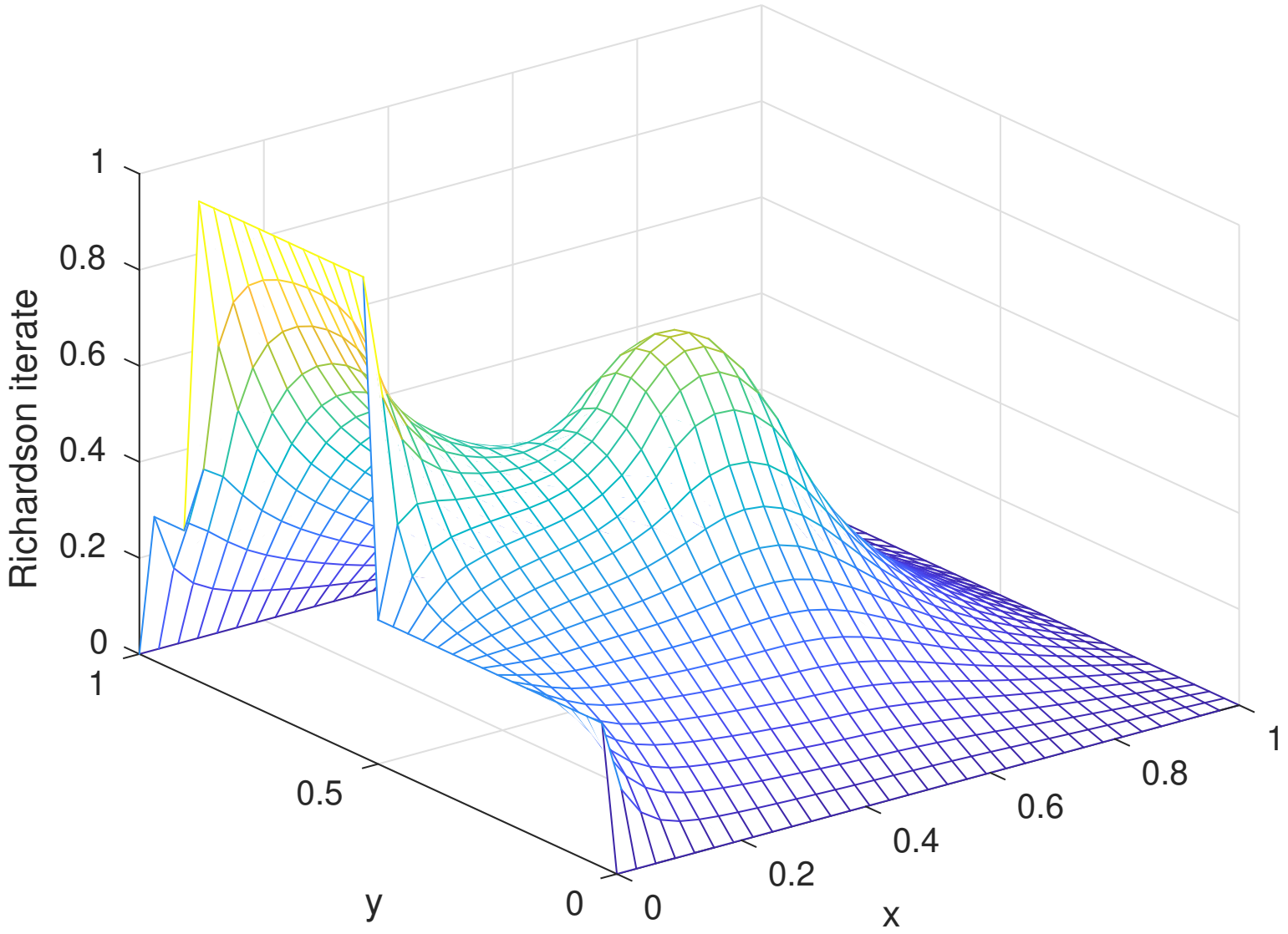
Richardson iteration 46



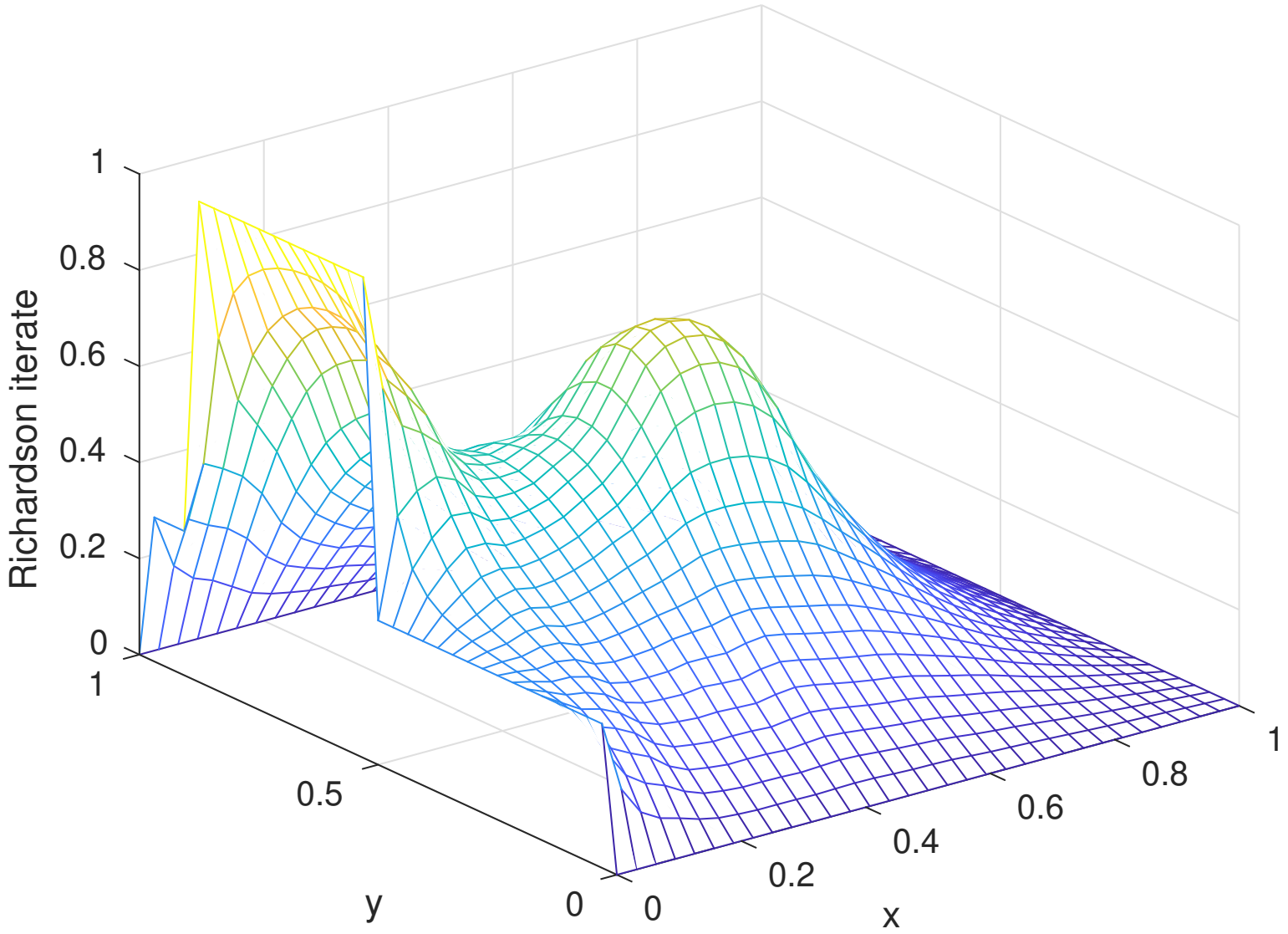
Richardson iteration 47



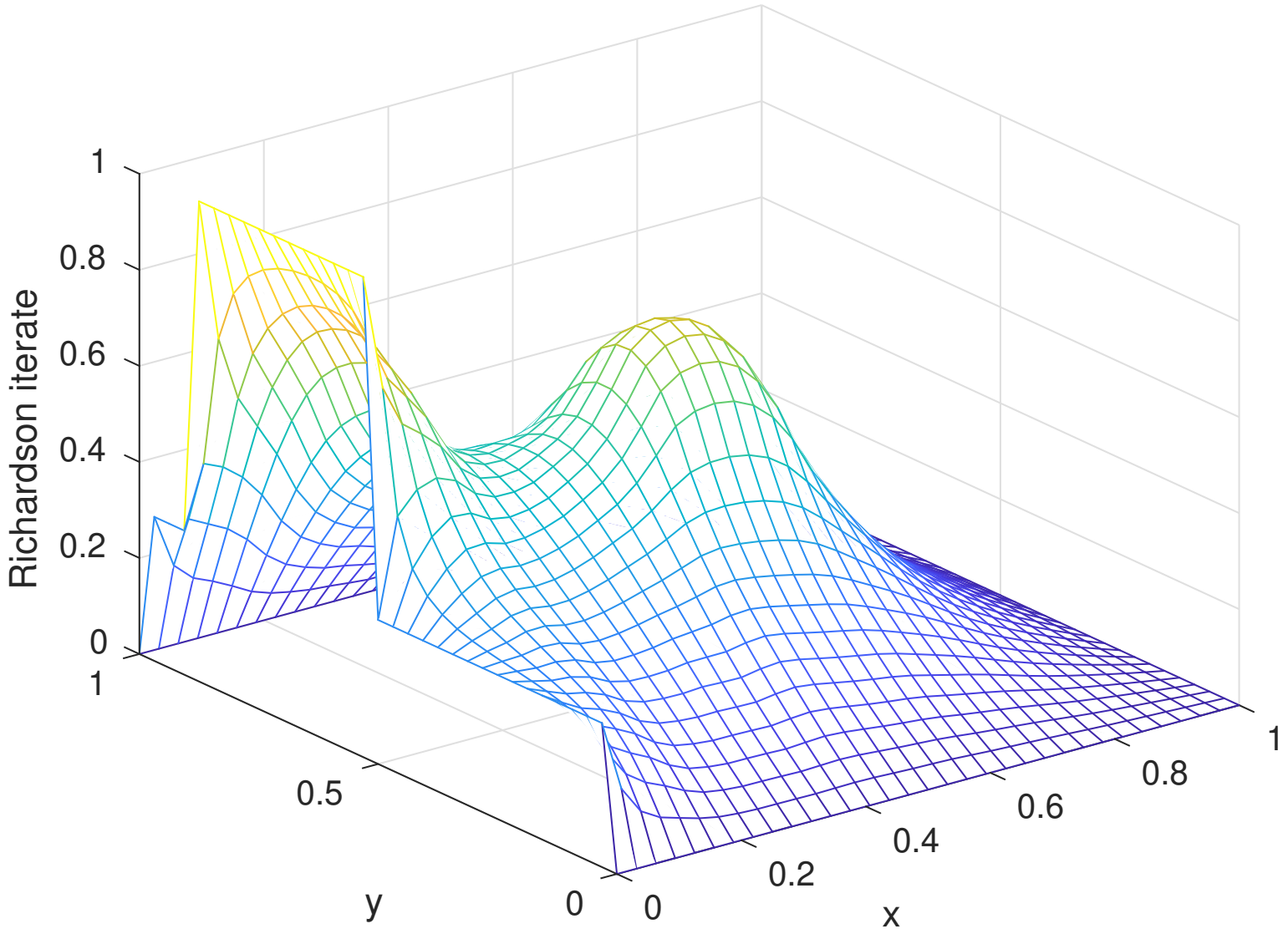
Richardson iteration 48



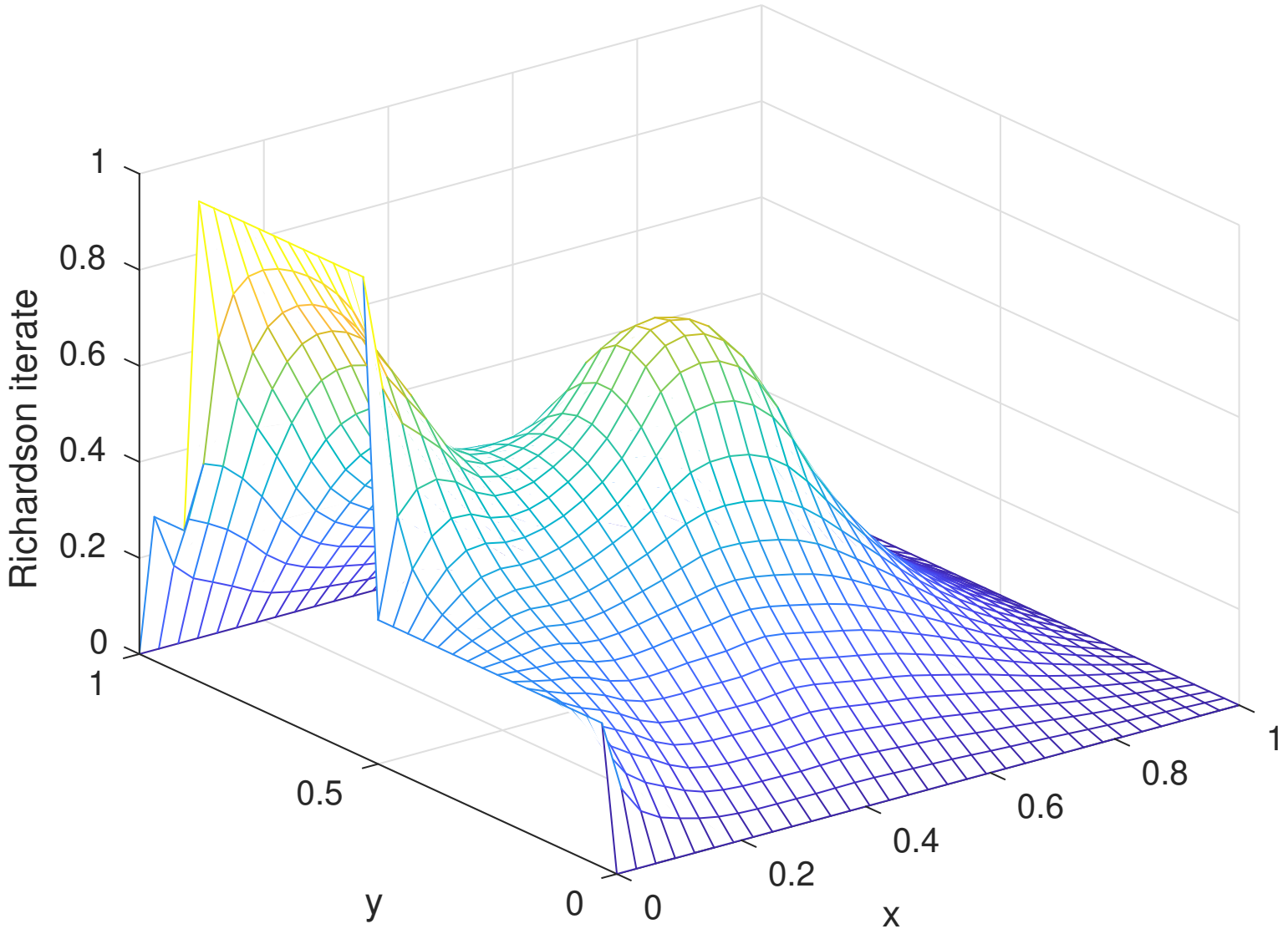
Richardson iteration 49



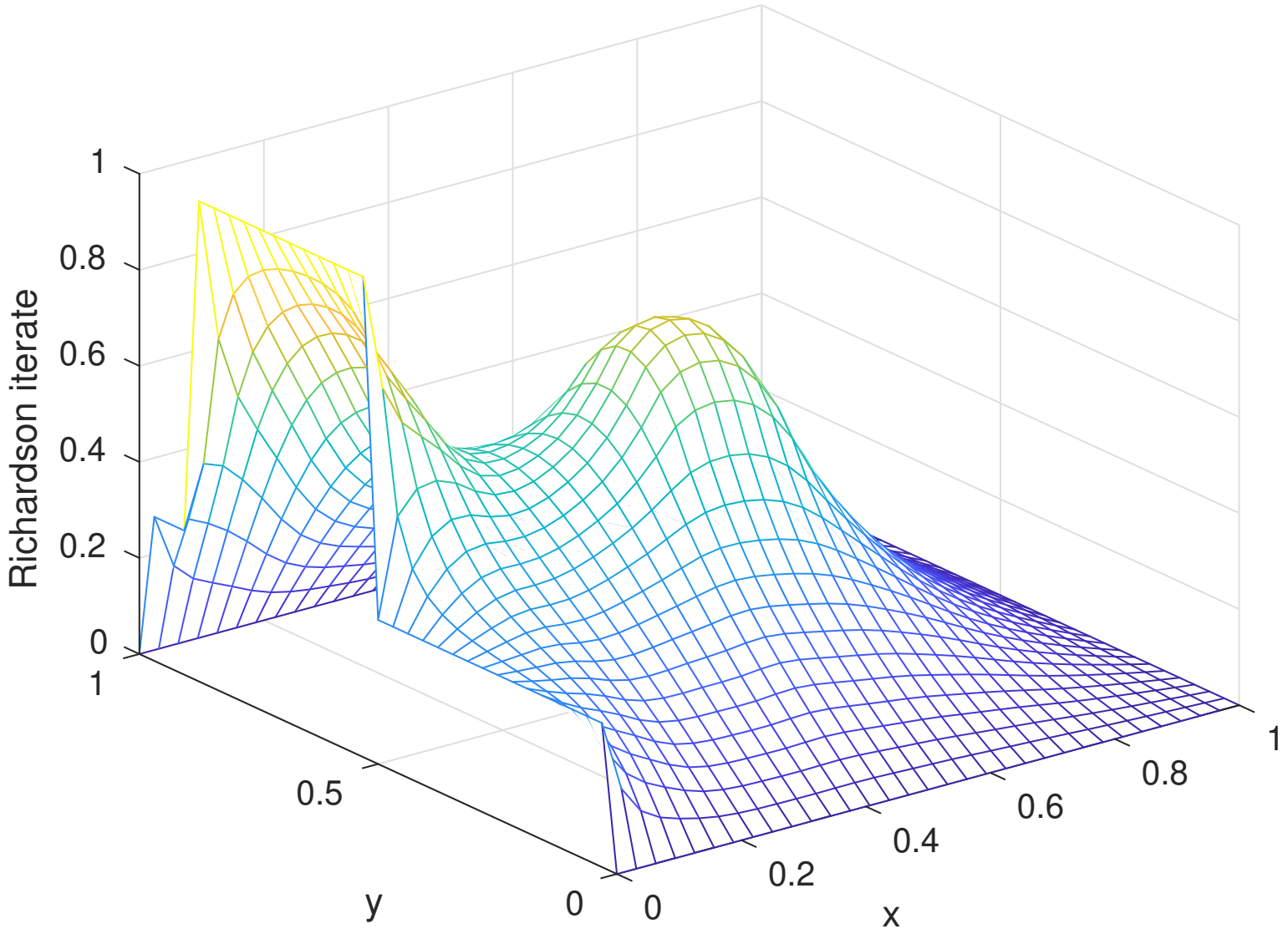
Richardson iteration 50



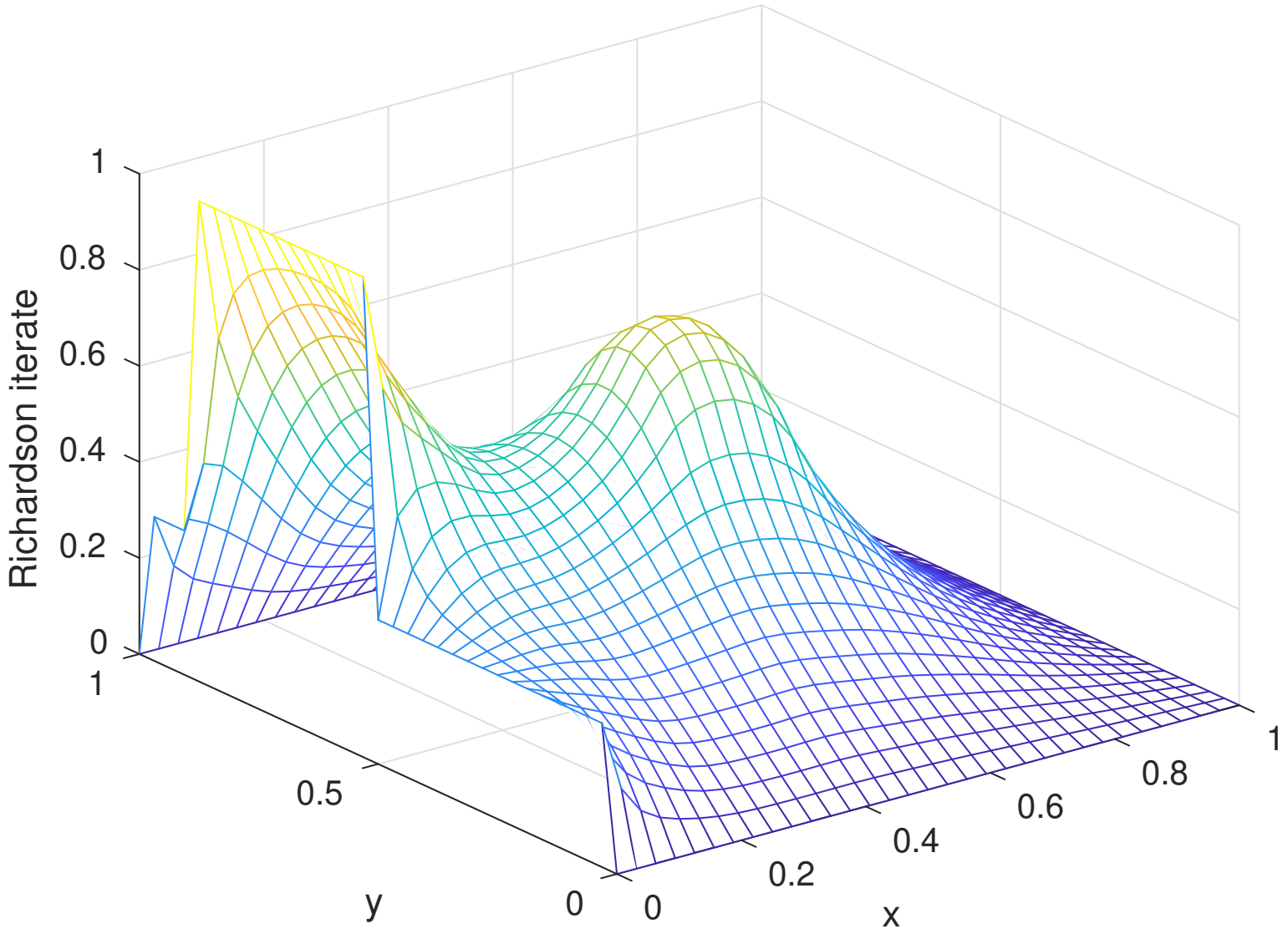
Richardson iteration 51



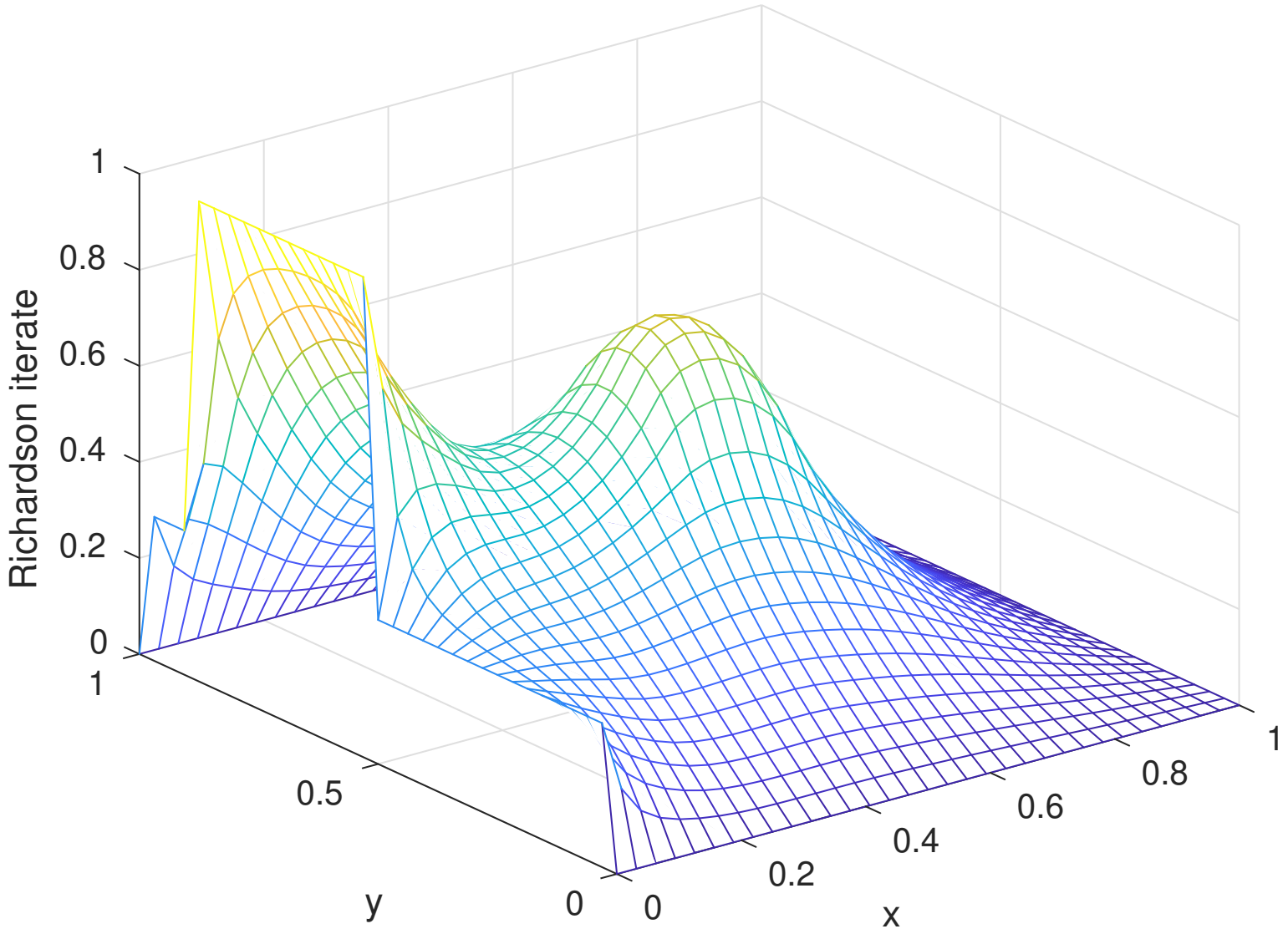
Richardson iteration 52



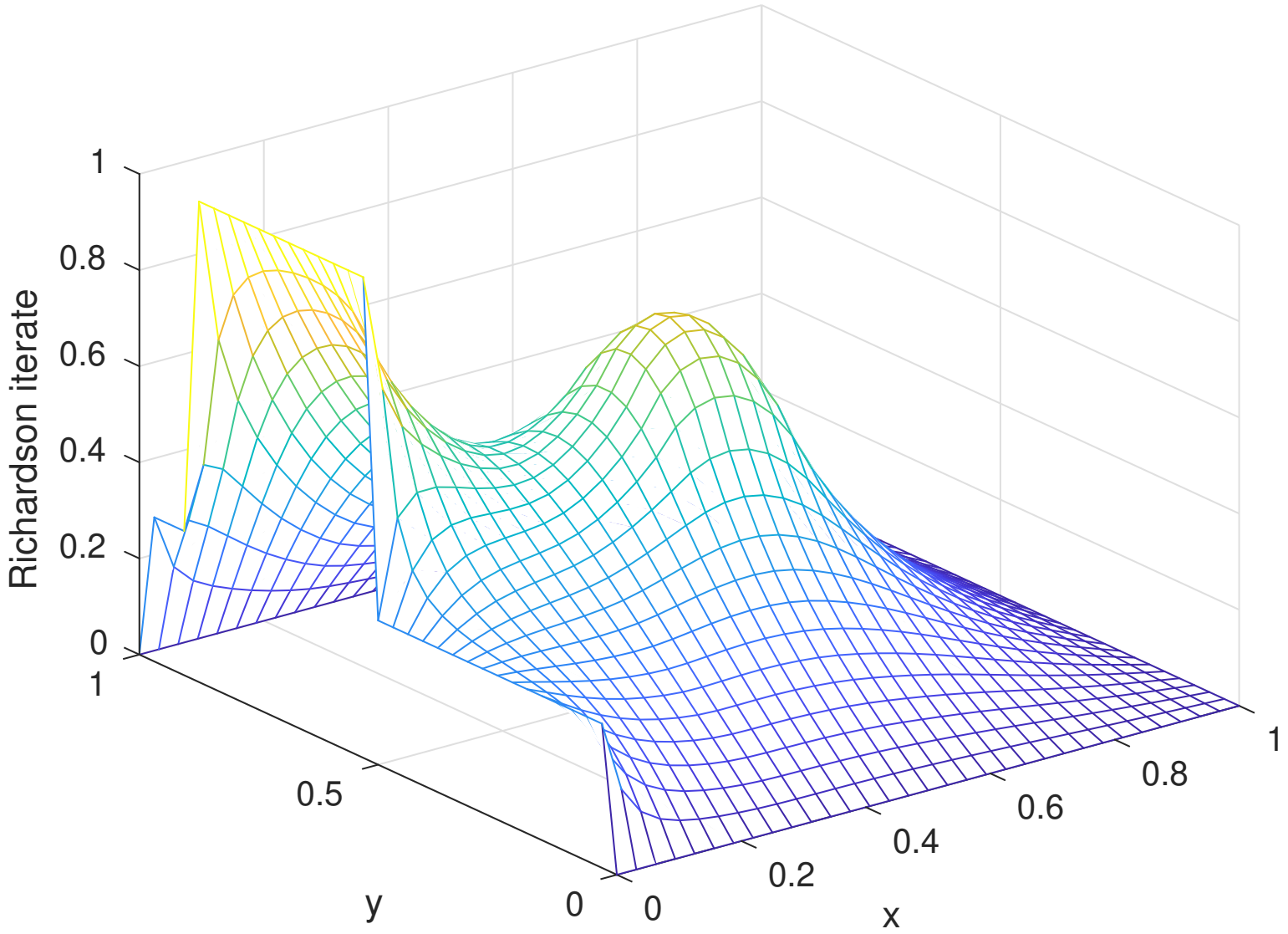
Richardson iteration 53



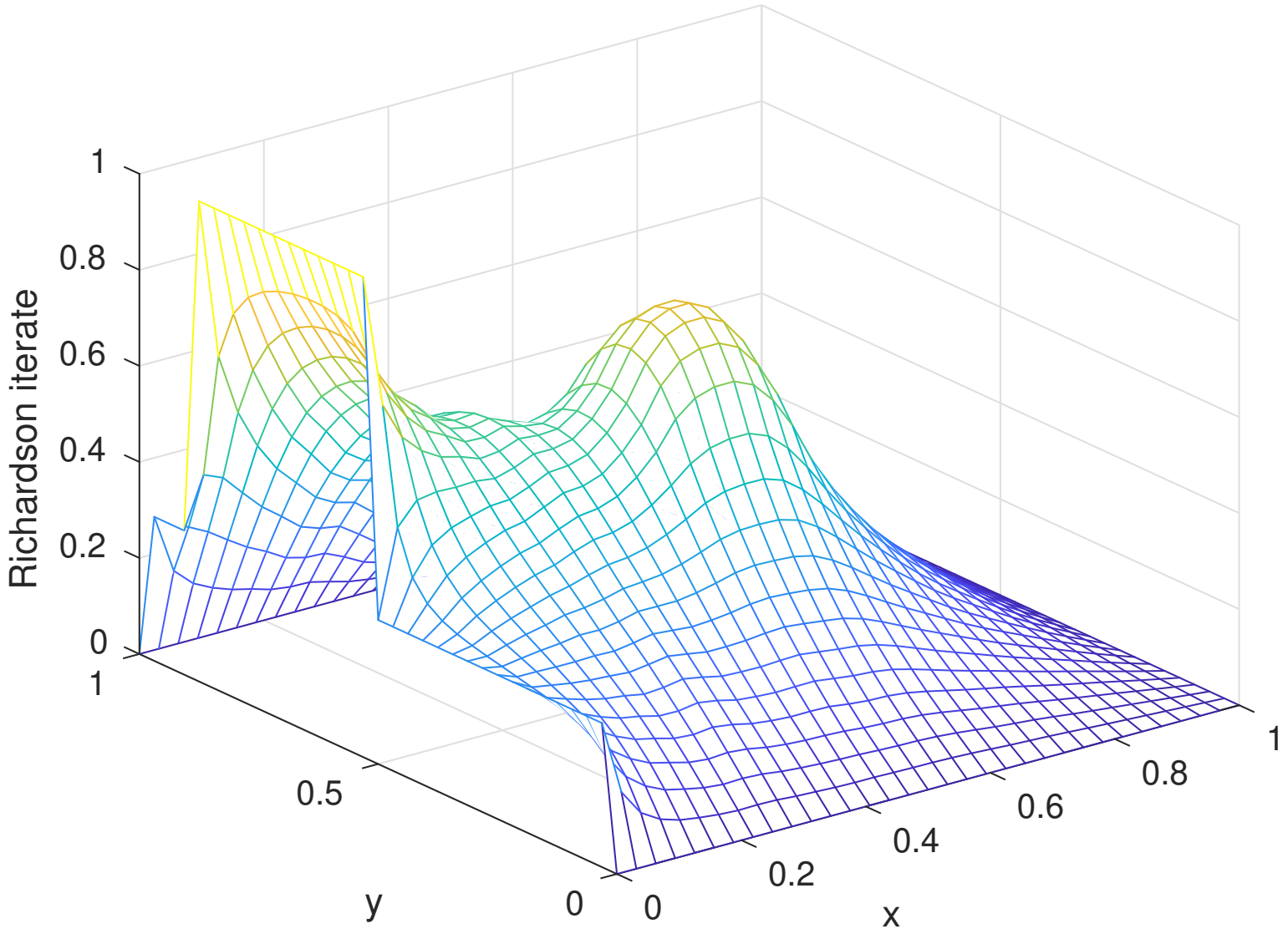
Richardson iteration 54



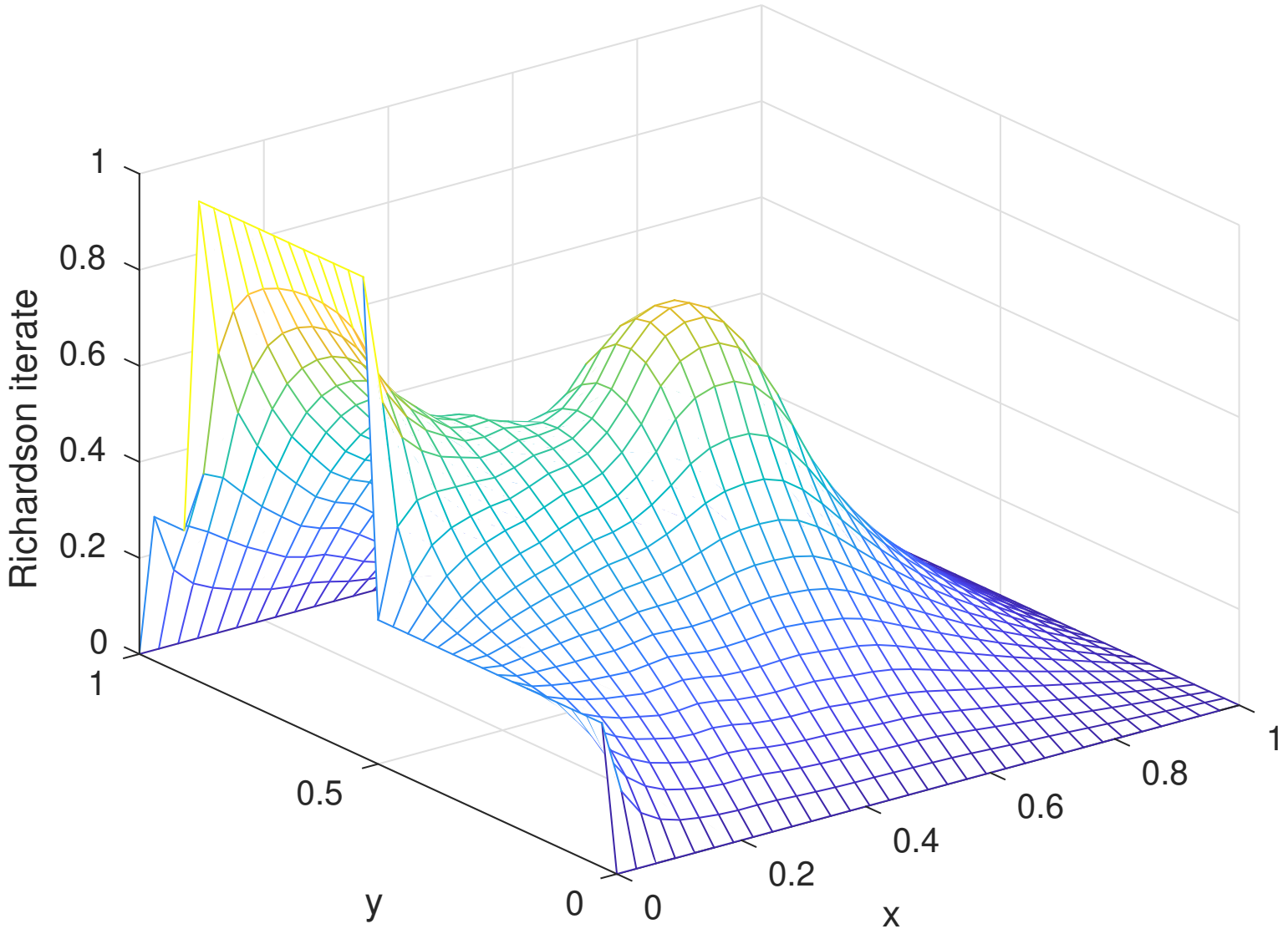
Richardson iteration 55



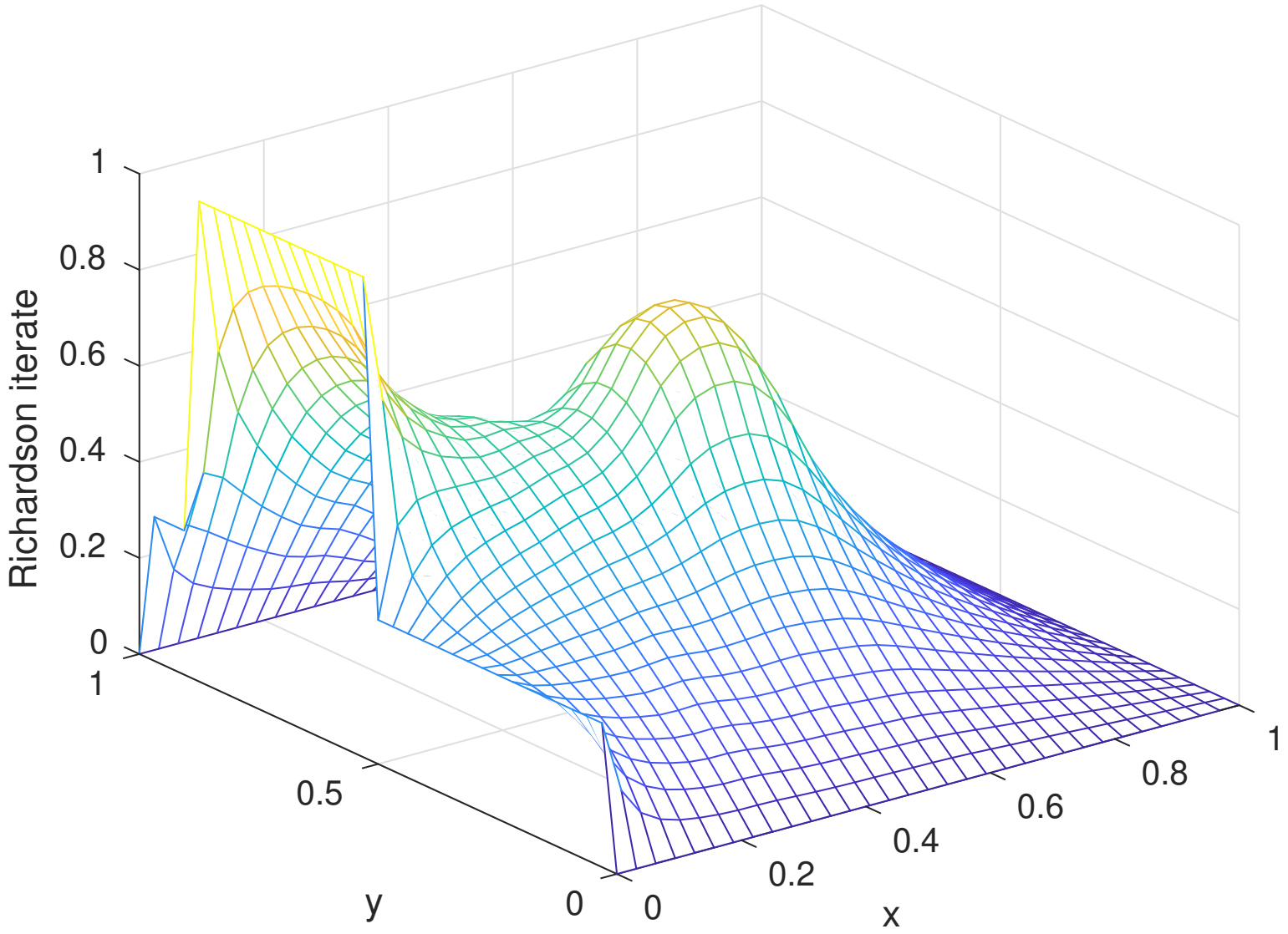
Richardson iteration 56



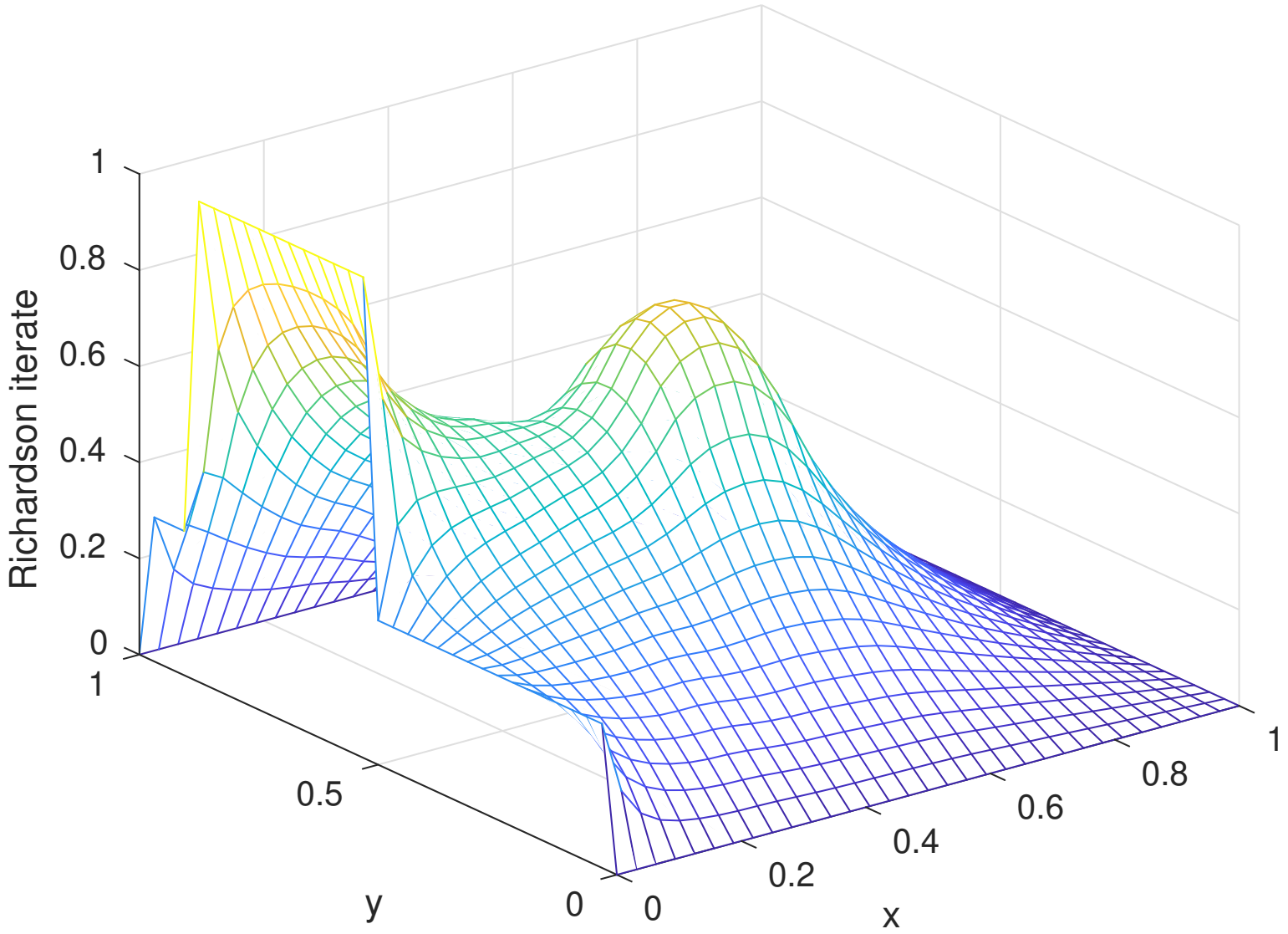
Richardson iteration 57



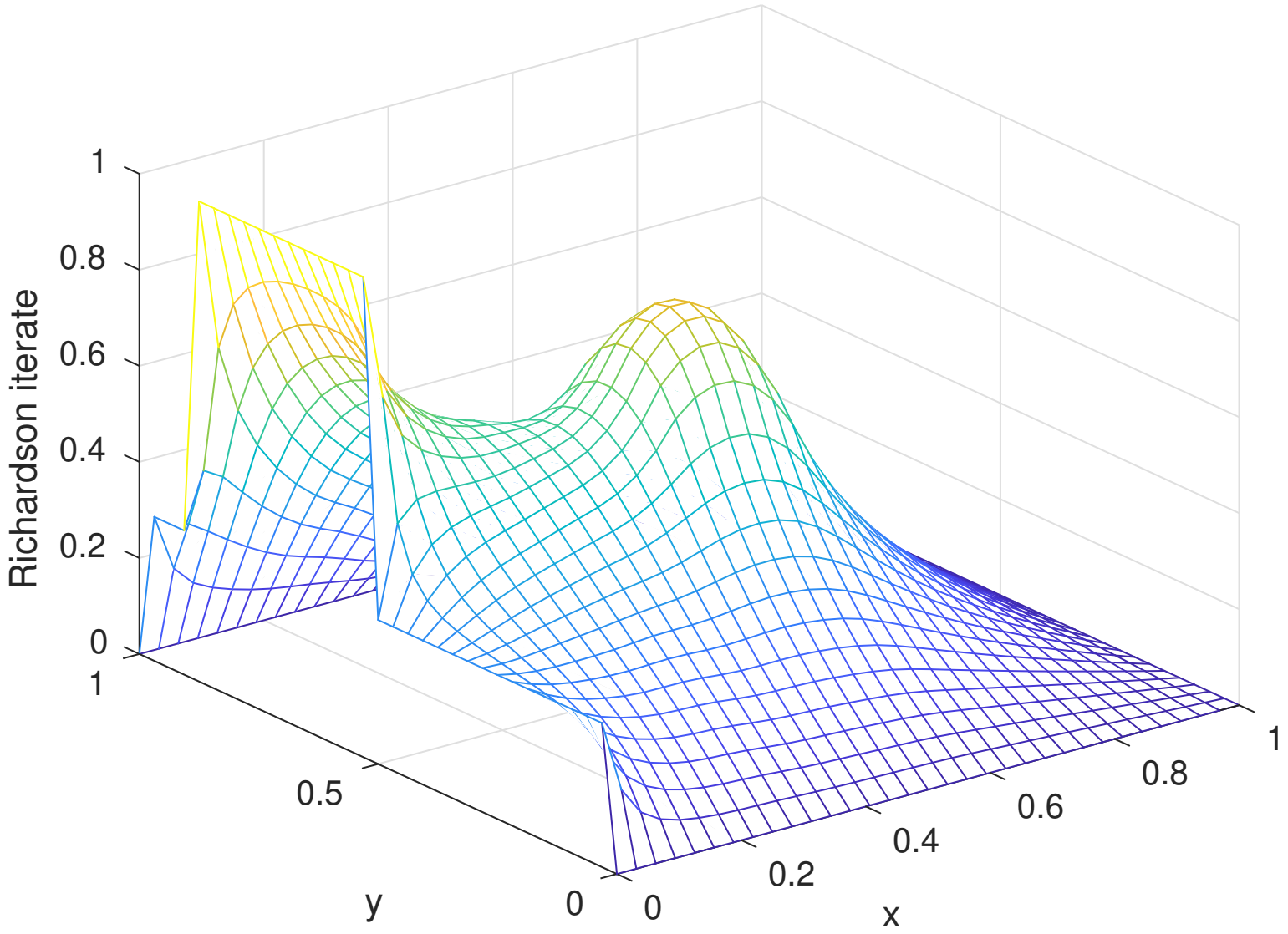
Richardson iteration 58



Richardson iteration 59



Richardson iteration 60



Development of Extrapolation Methods

- **Gene Golub's (1959)** PhD-thesis: the Chebyshev Semi-Iterative Method!
- **Claude Brezinski (1977):** Accélération de la Convergence en Analyse Numérique (Springer)
- **Avram Sidi and Jacob Bridger (1988):** Convergence and stability analyses for some vector extrapolation methods in the presence of defective iteration matrices
- **Avram Sidi (1991):** Efficient implementation of minimal polynomial and reduced rank extrapolation methods

Introduction in W. Gander, M.J. Gander, F. Kwok (2014): Scientific Computing: An introduction Using Maple and Matlab.

Iterative Methods

Martin J. Gander

Stationary Methods
Gauss
Jacobi, Seidel
Example
Modern Notation
SOR

Extrapolation
Richardson
Example
Young, Golub, Bresinski, Sidi

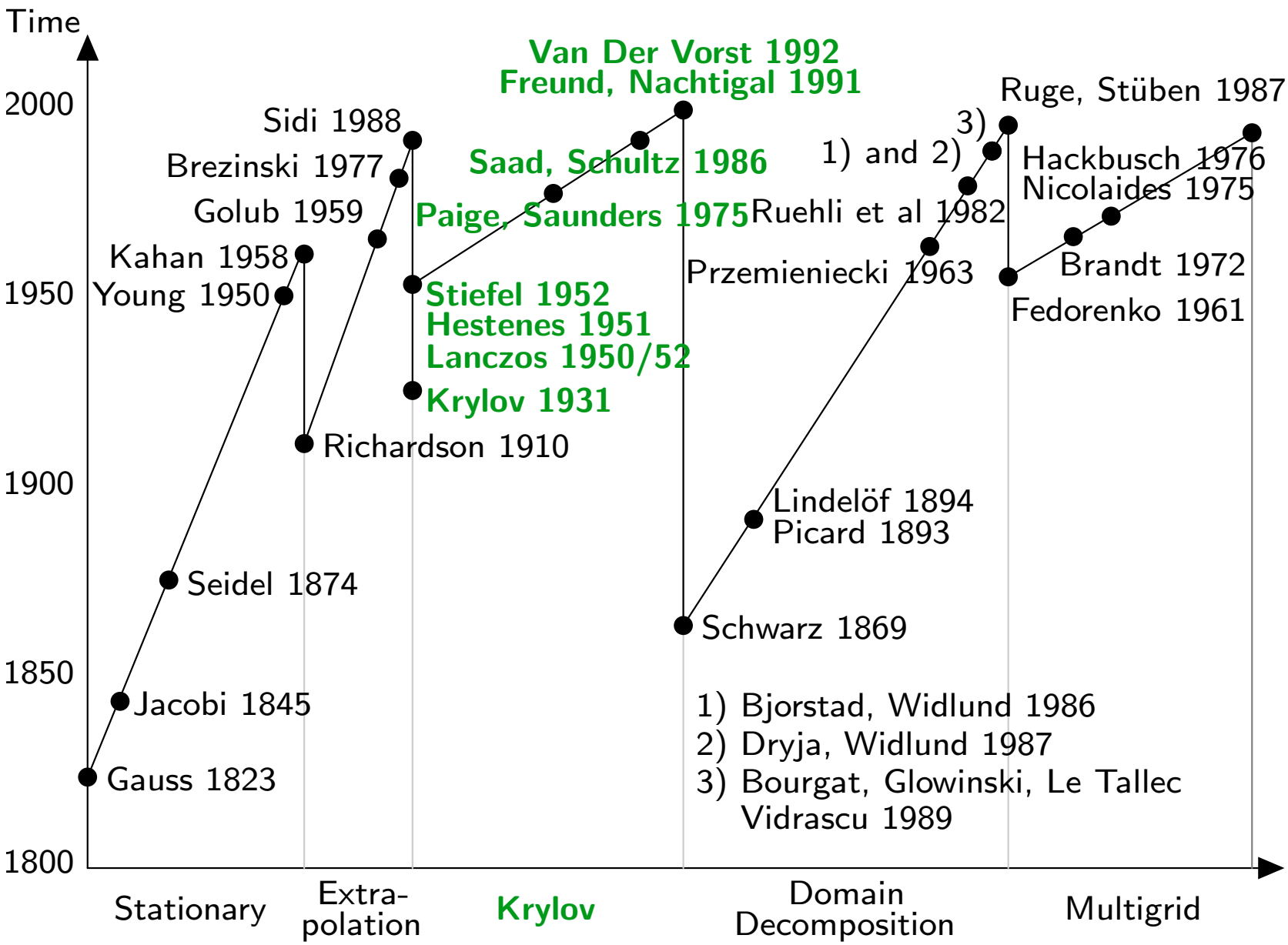
Krylov Methods
Conjugate Gradients
Example
Paige, Saunders, Saad, Freund, Van Der Vorst

Domain Decomposition
Invention of Schwarz
Schwarz example
Further DD Methods
Example
Optimal Schwarz

Multigrid
MG example
Fedorenko

Preconditioning

Krylov Methods for Linear Systems



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Martin J. Gander

Stationary Methods
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Fedorenko

Preconditioning

Invention of Conjugate Gradients

Stiefel and Rosser 1951: Presentations at a Symposium at the National Bureau of Standards (UCLA)

Hestenes 1951: Iterative methods for solving linear equations

Stiefel 1952: Über einige Methoden der Relaxationsrechnung

Hestenes and Stiefel 1952: Methods of Conjugate Gradients for Solving Linear Systems

"An iterative algorithm is given for solving a system $Ax = k$ of n linear equations in n unknowns. The solution is given in n steps."

Lanczos 1952: Solution of systems of linear equations by minimized iterations (see also 1950)

Stationary Methods
- Gauss
- Jacobi, Seidel
- Example
- Modern Notation
- SOR

Extrapolation
- Richardson
- Example
- Young, Golub, Bresinski, Sidi

Krylov Methods
- Conjugate Gradients
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- Invention of Schwarz
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- Example
- Optimal Schwarz

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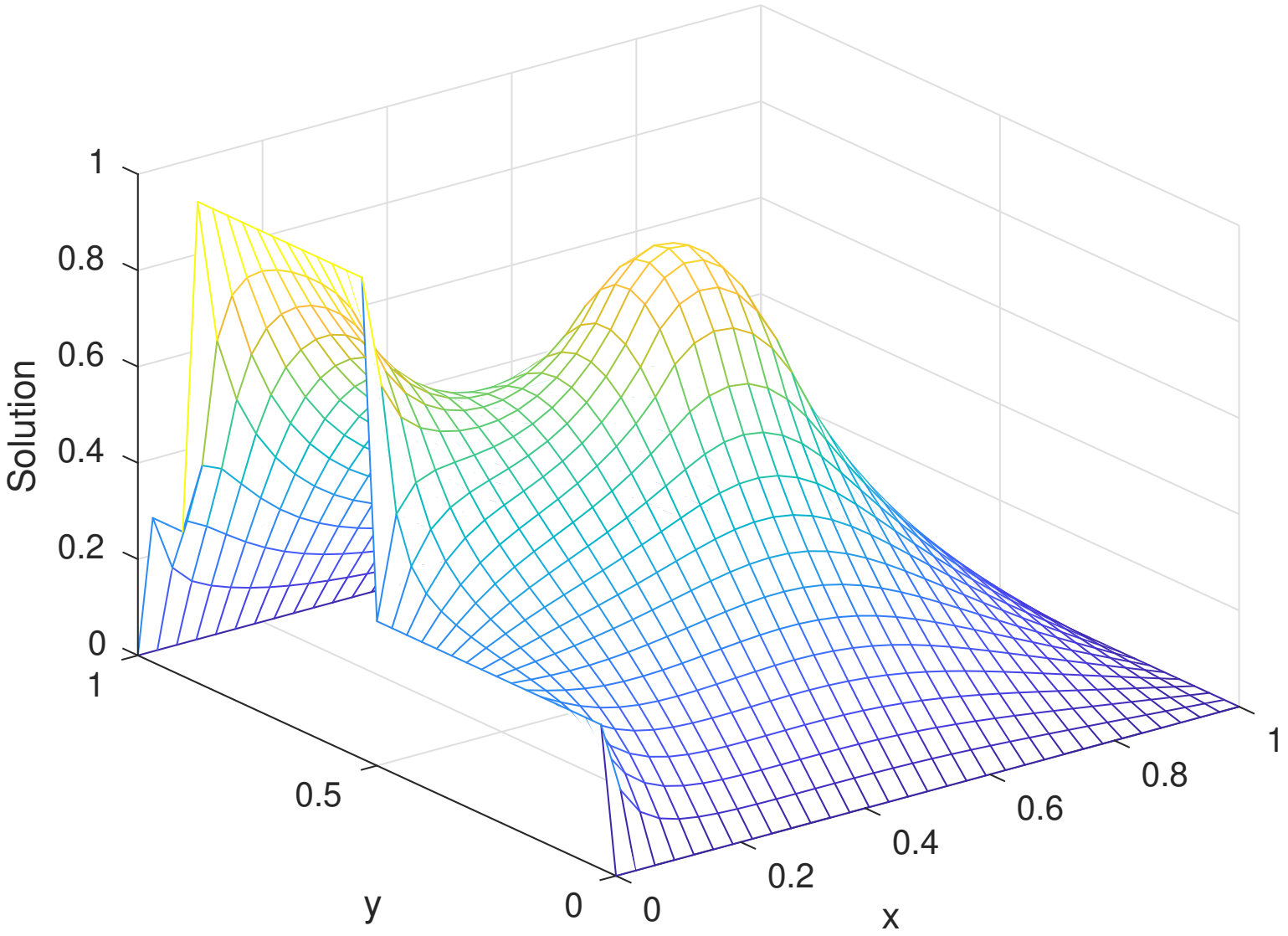
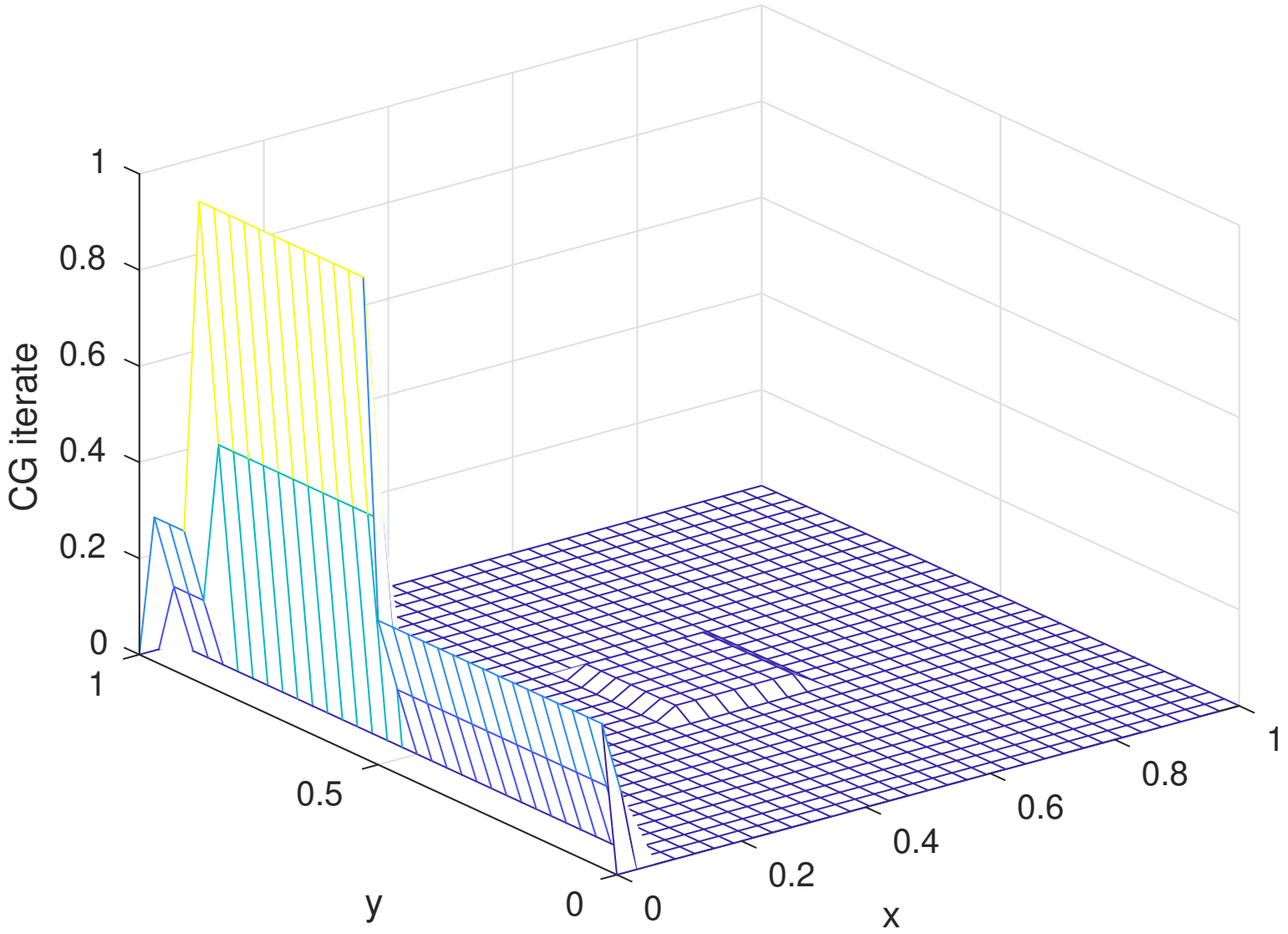
- MG example
- Fedorenko

Preconditioning

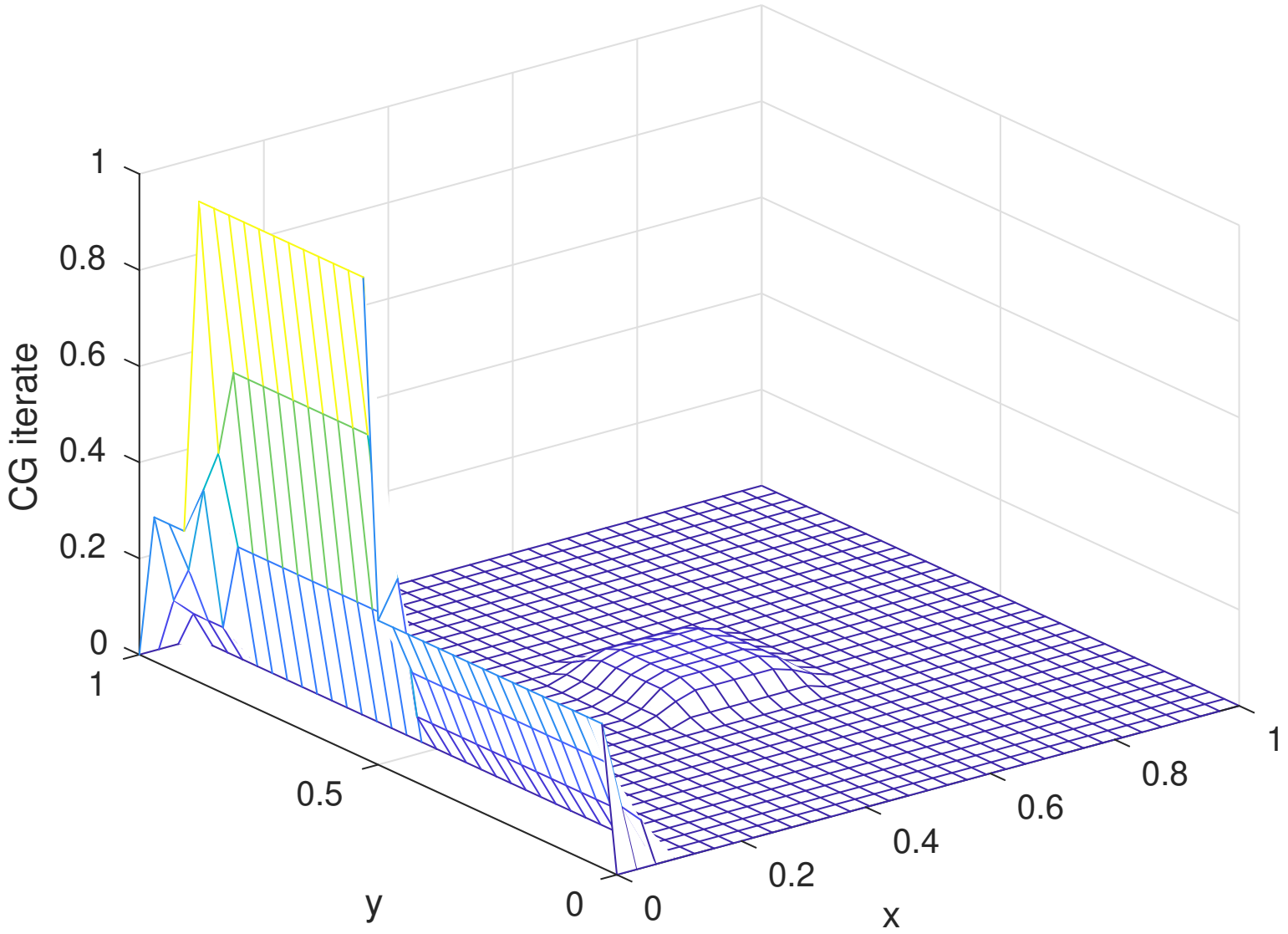
Example: temperature in a room



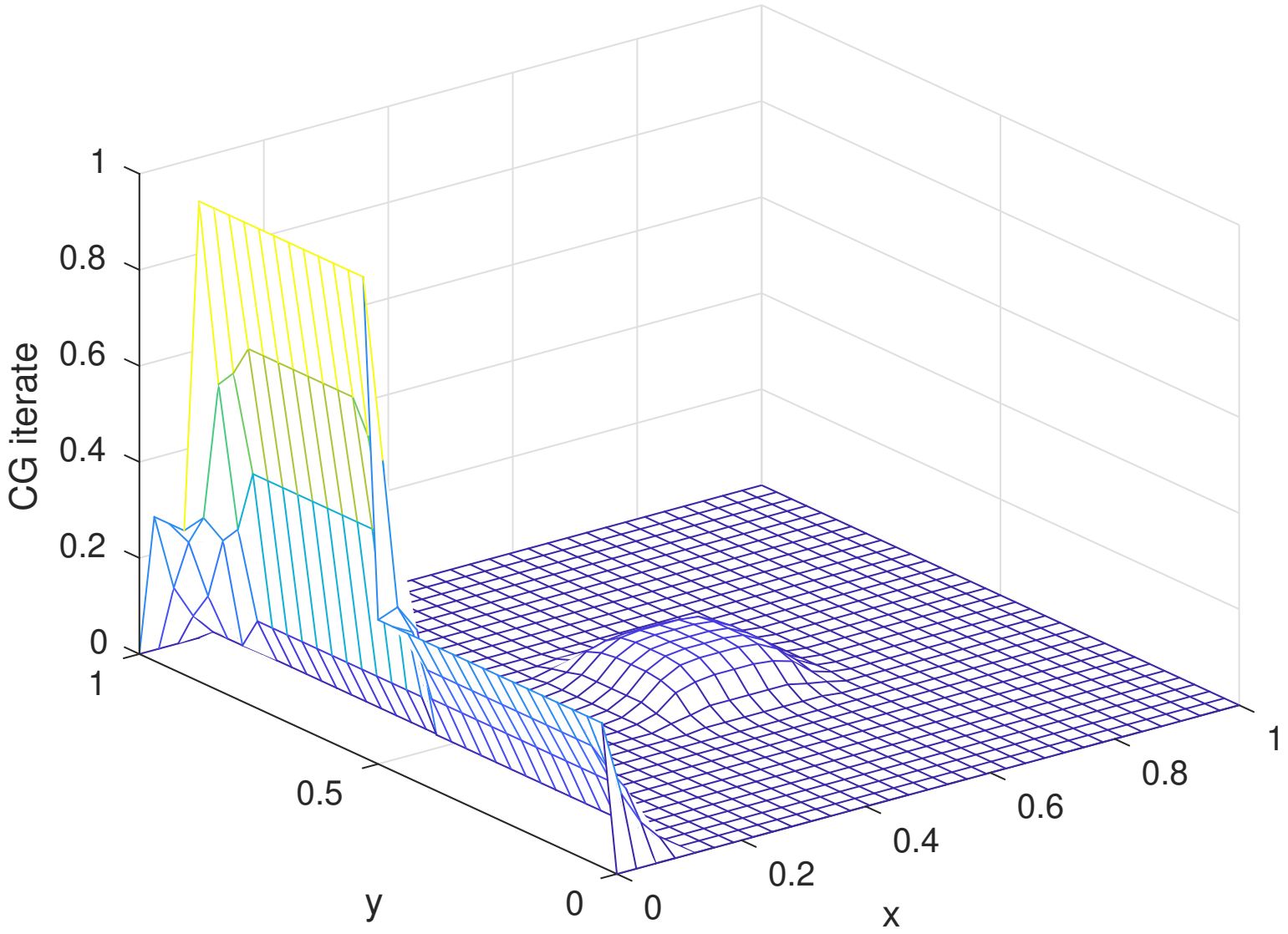
CG iteration 1



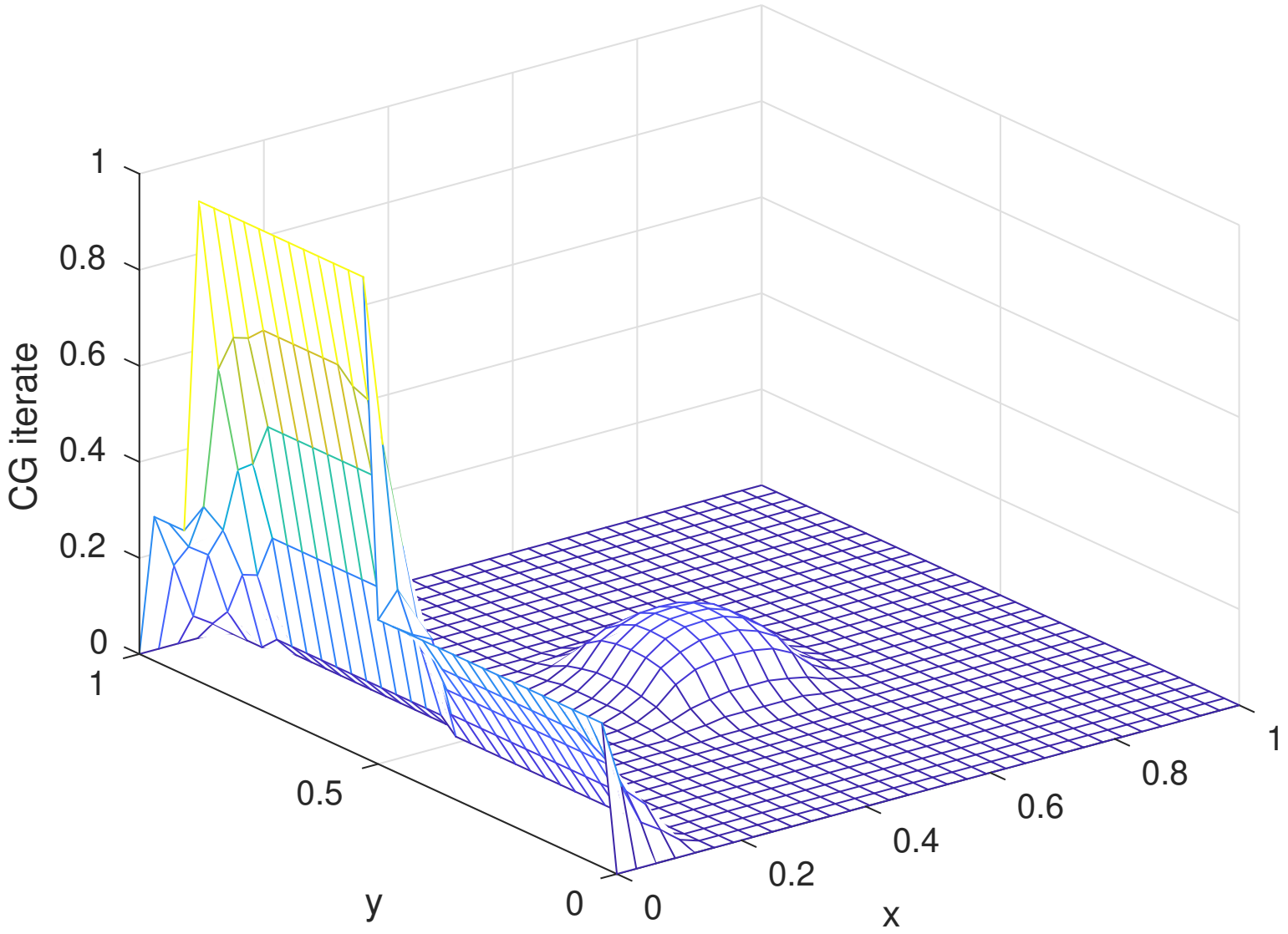
CG iteration 2



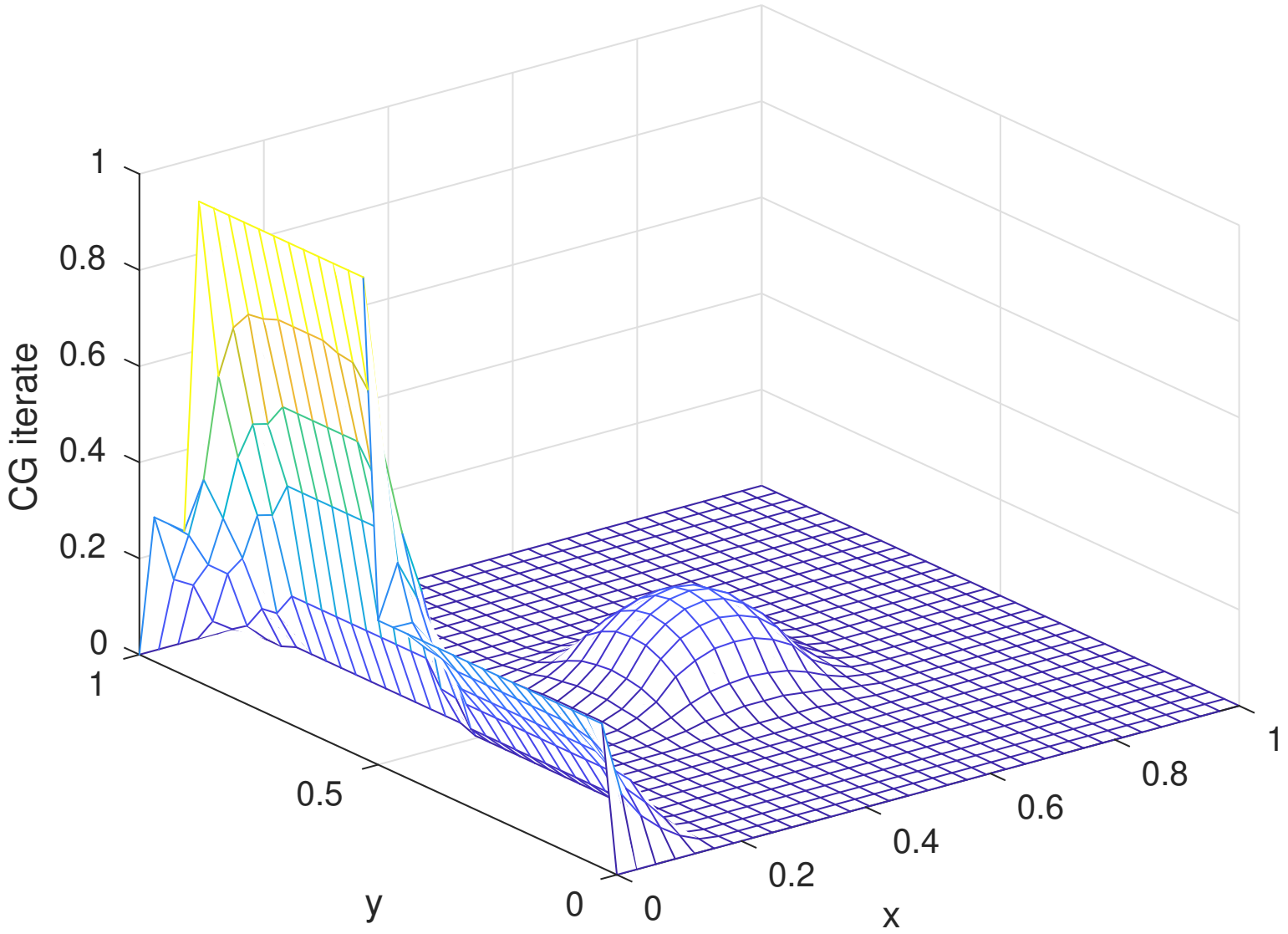
CG iteration 3



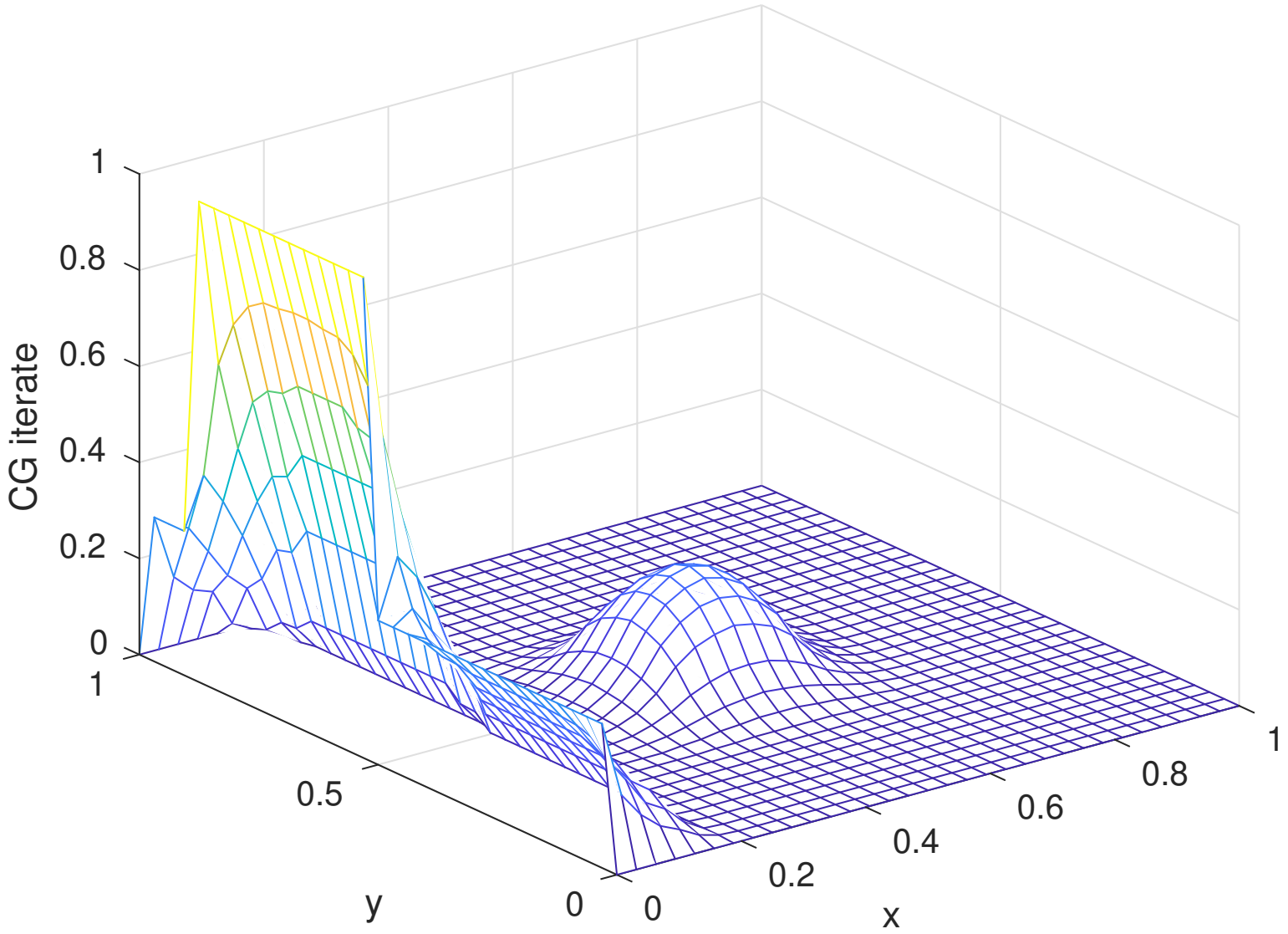
CG iteration 4



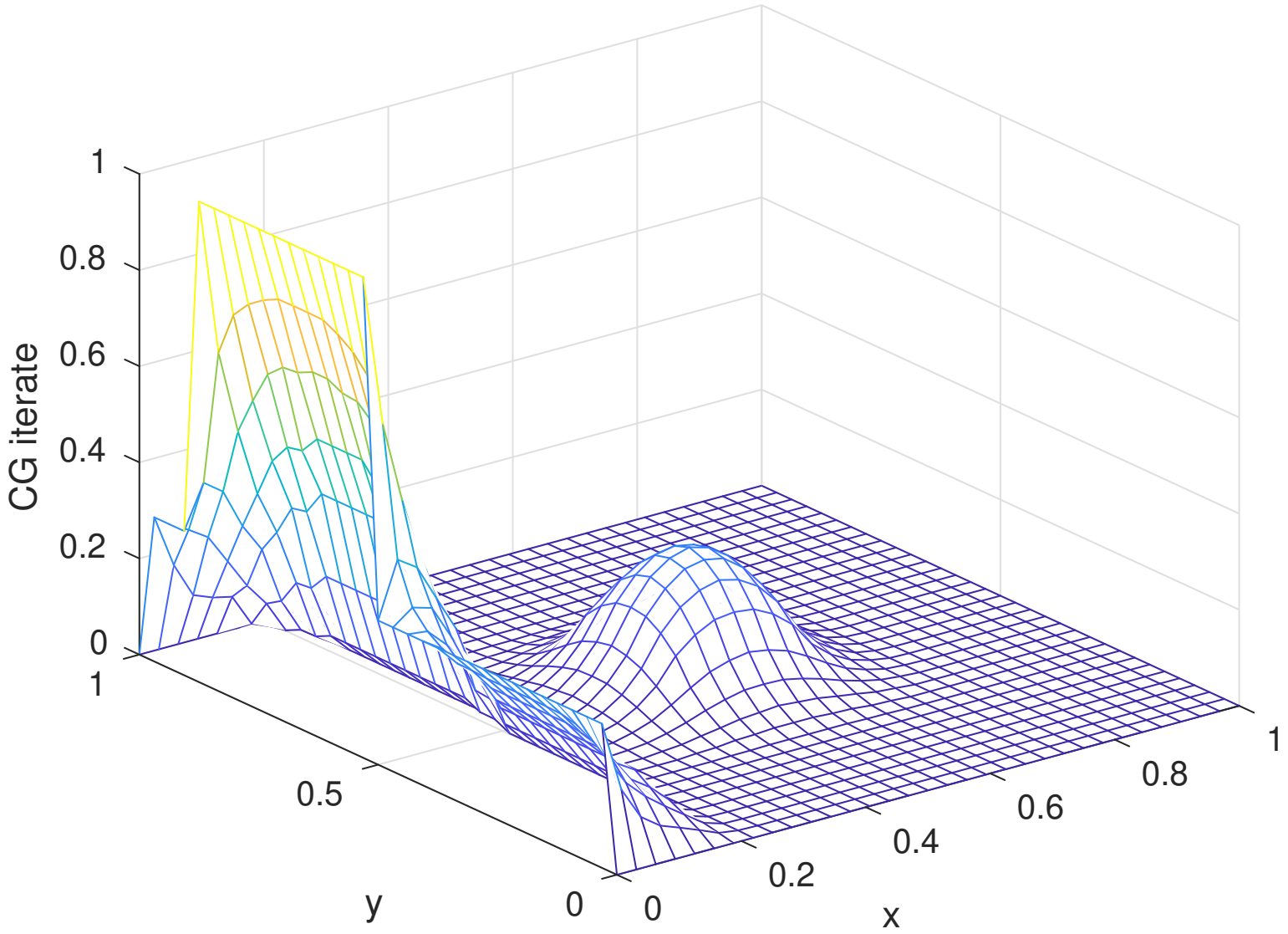
CG iteration 5



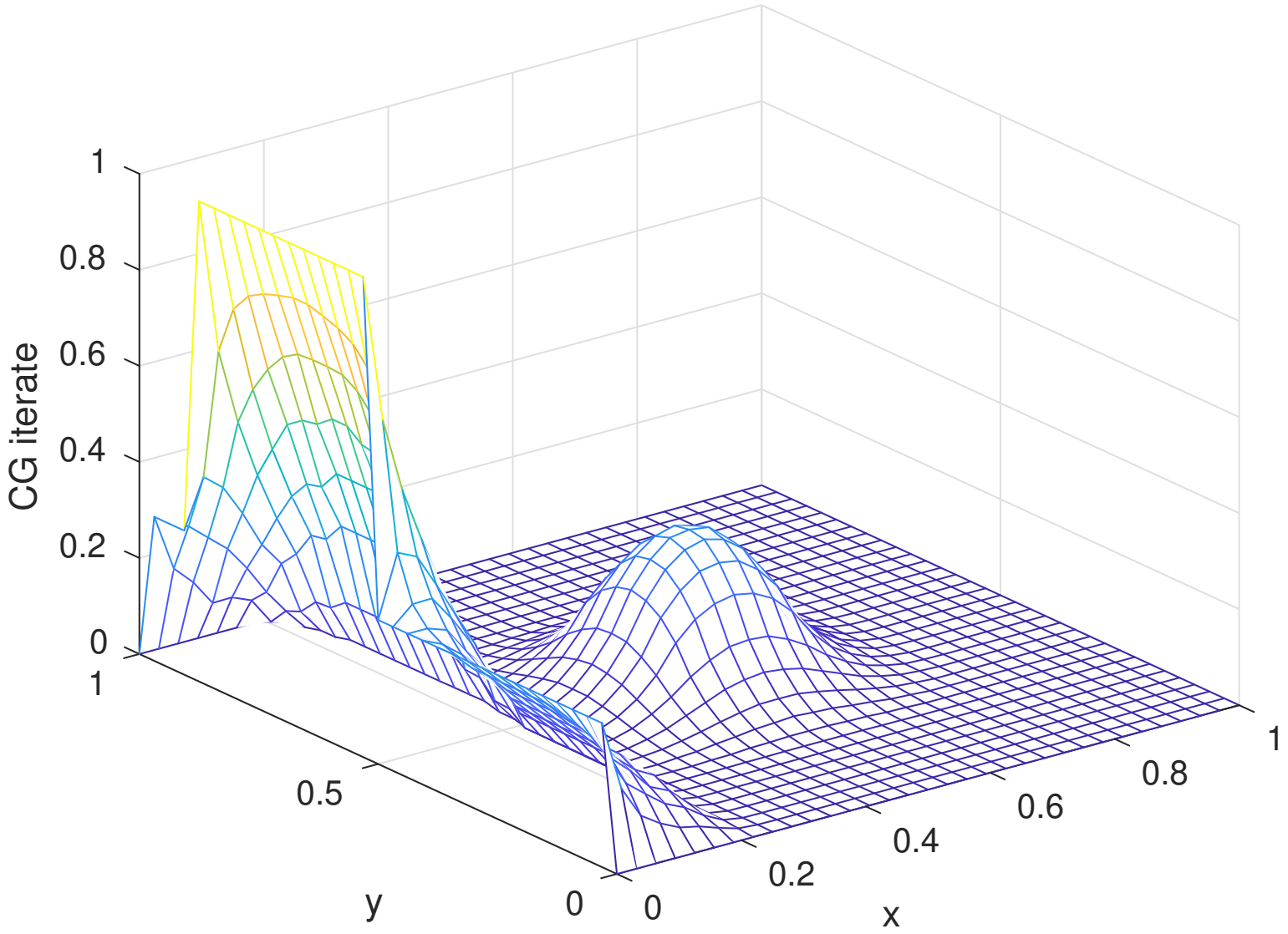
CG iteration 6



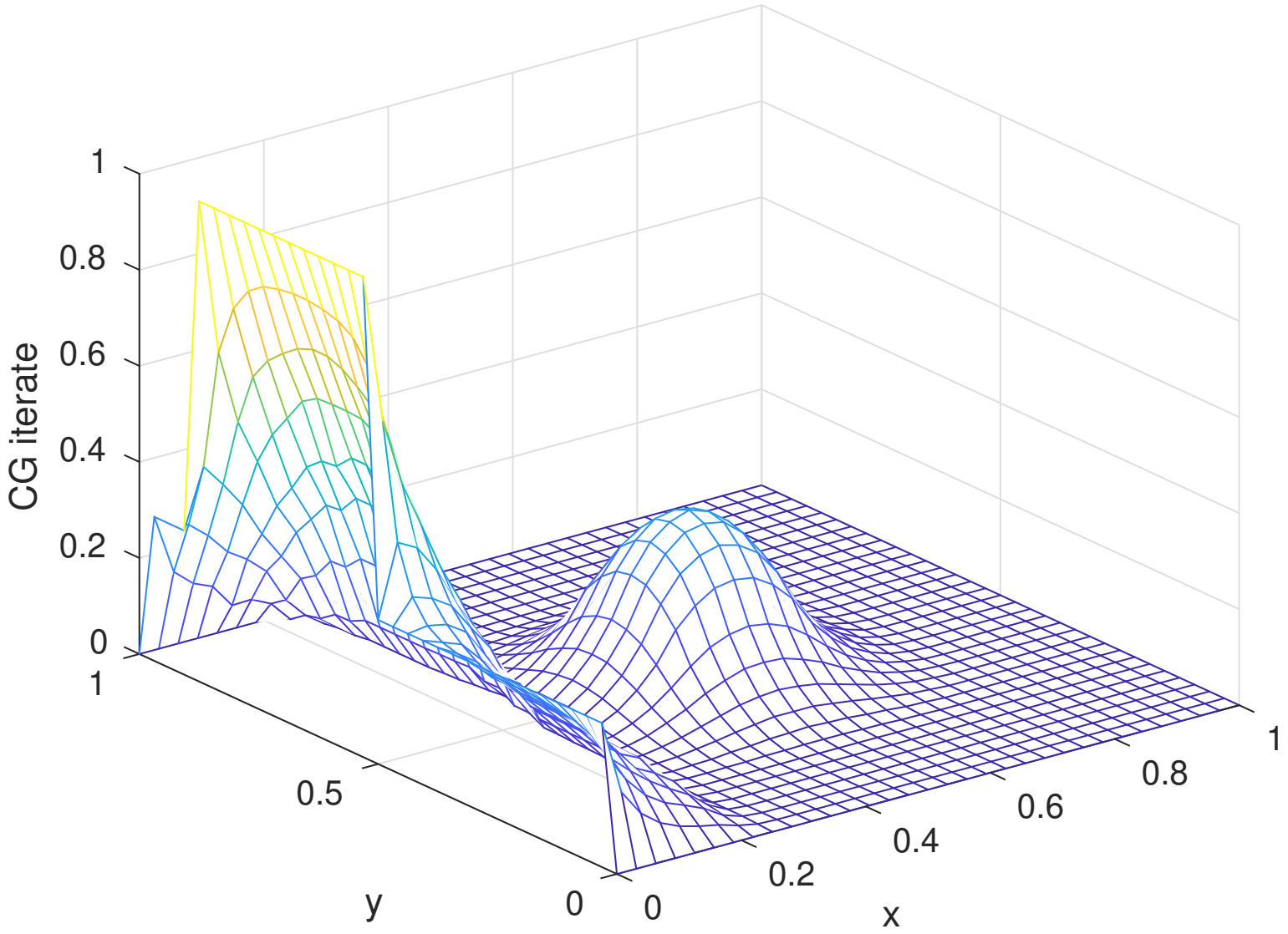
CG iteration 7



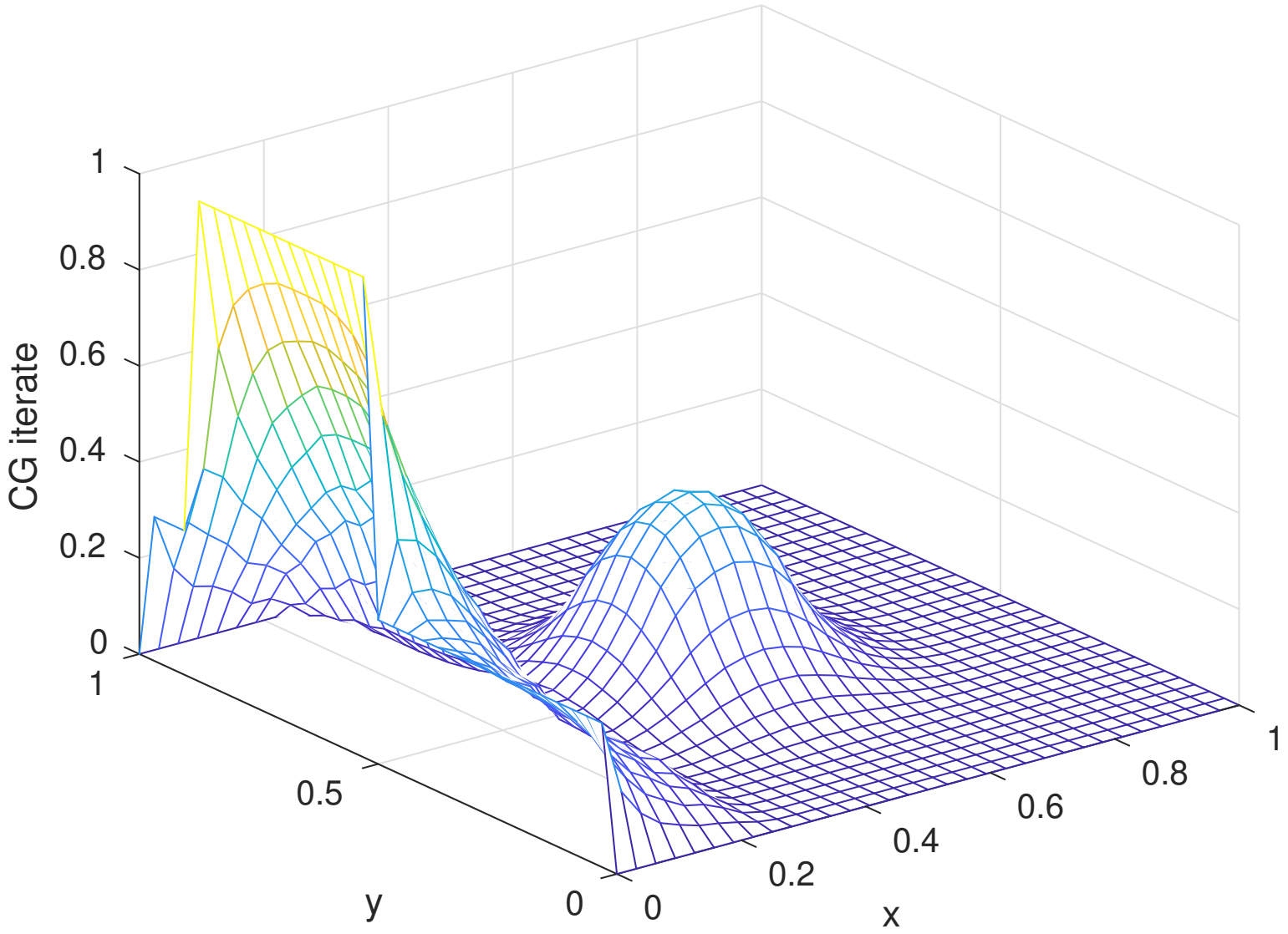
CG iteration 8



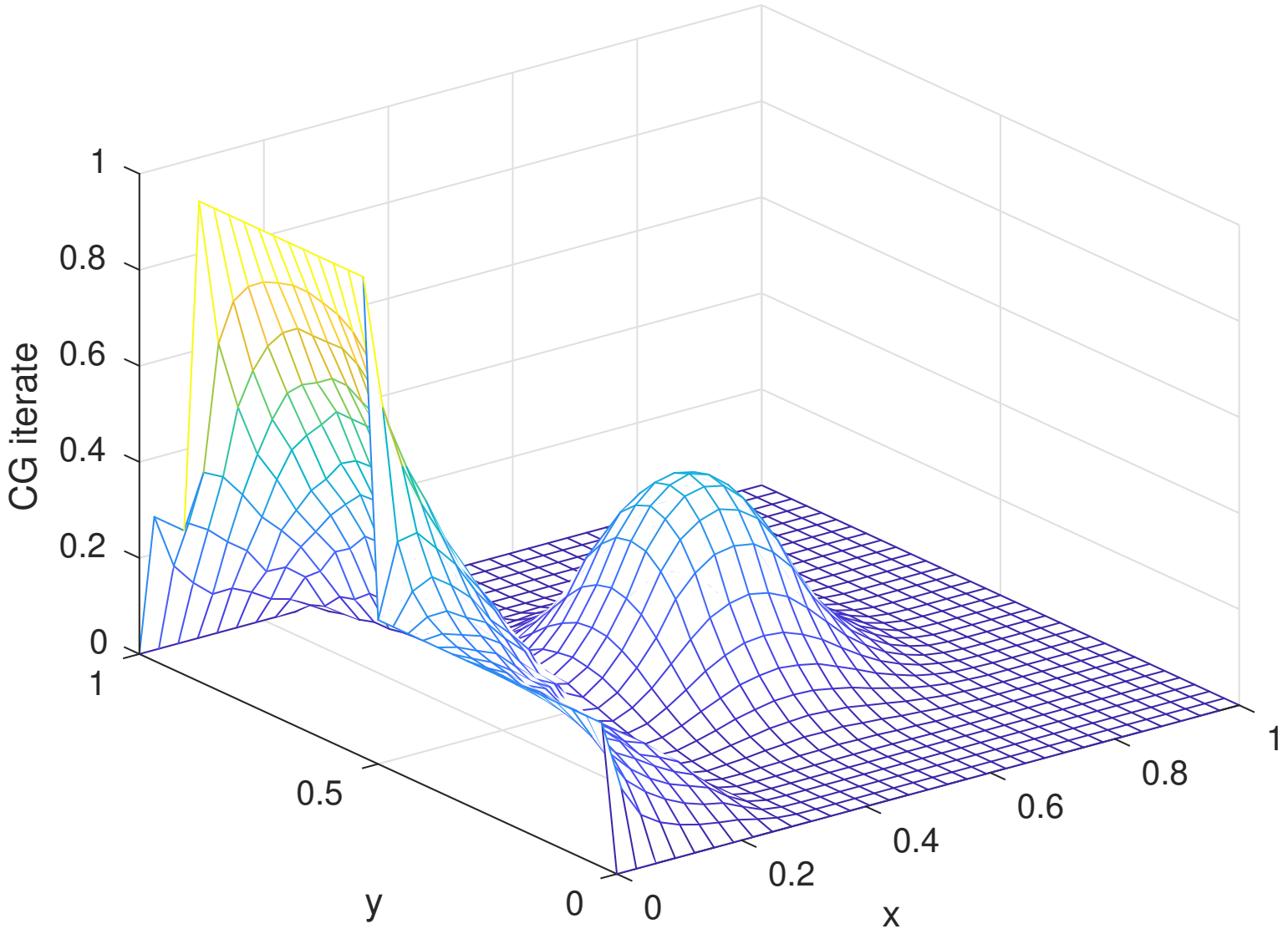
CG iteration 9



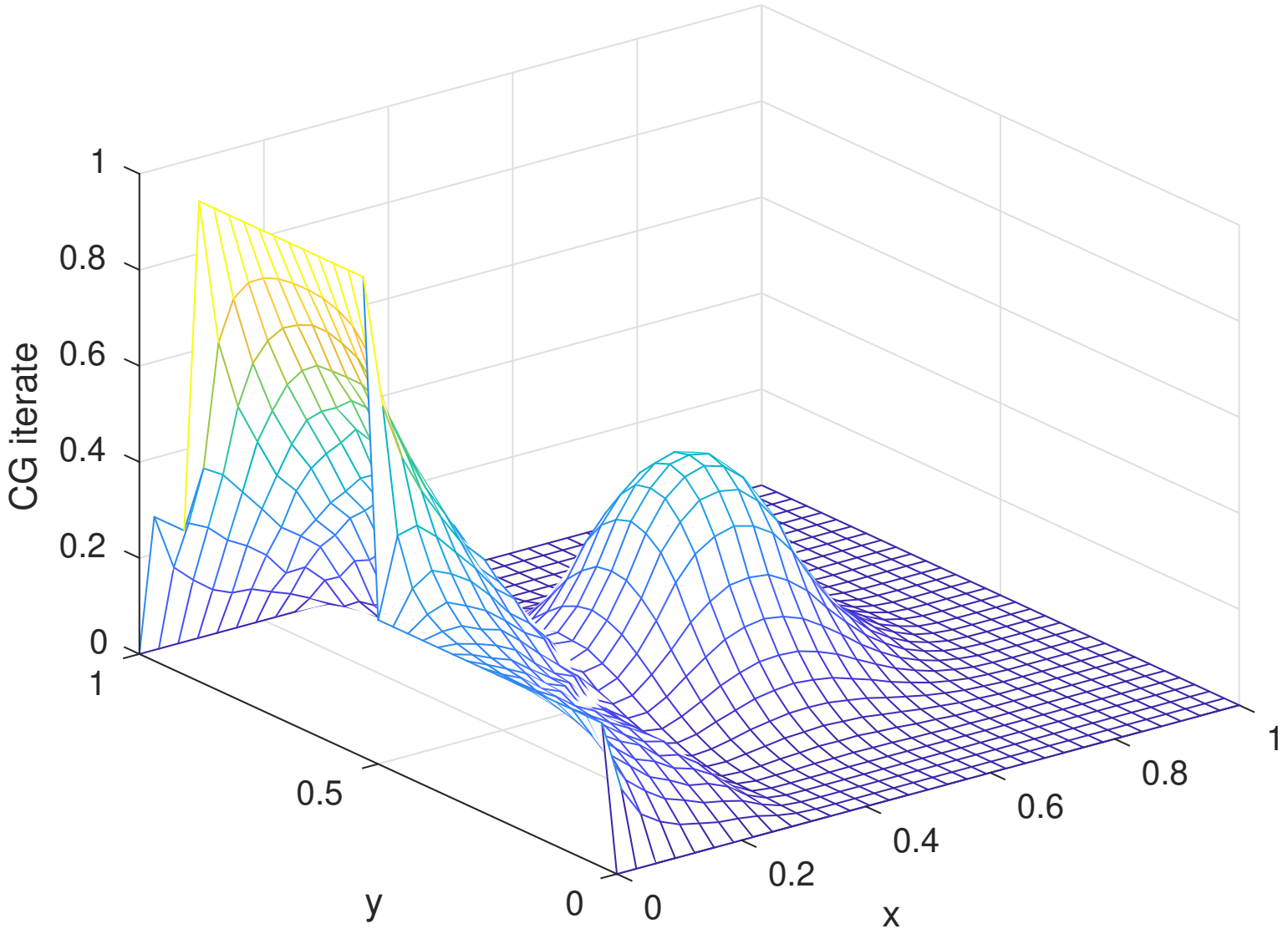
CG iteration 10



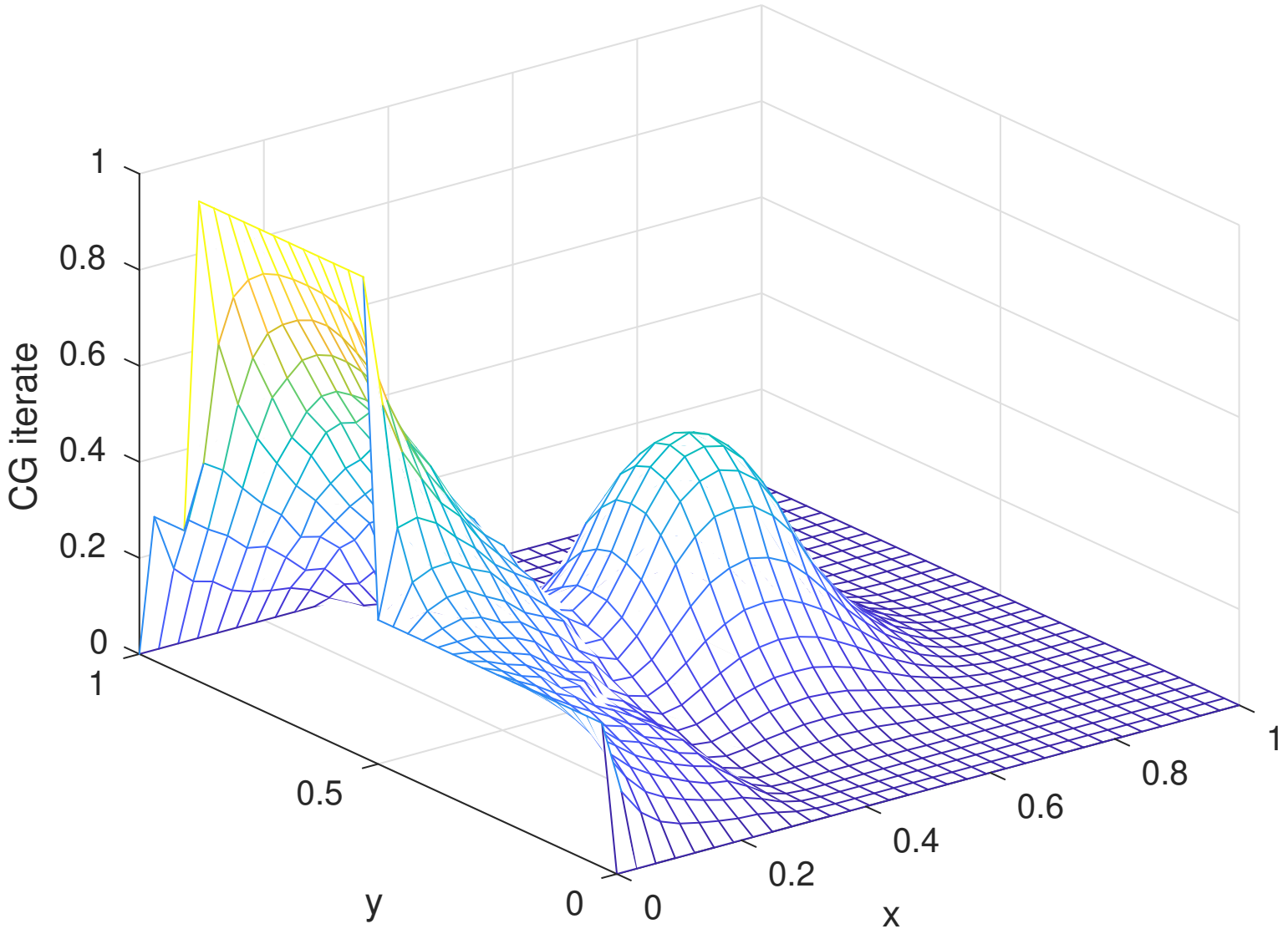
CG iteration 11



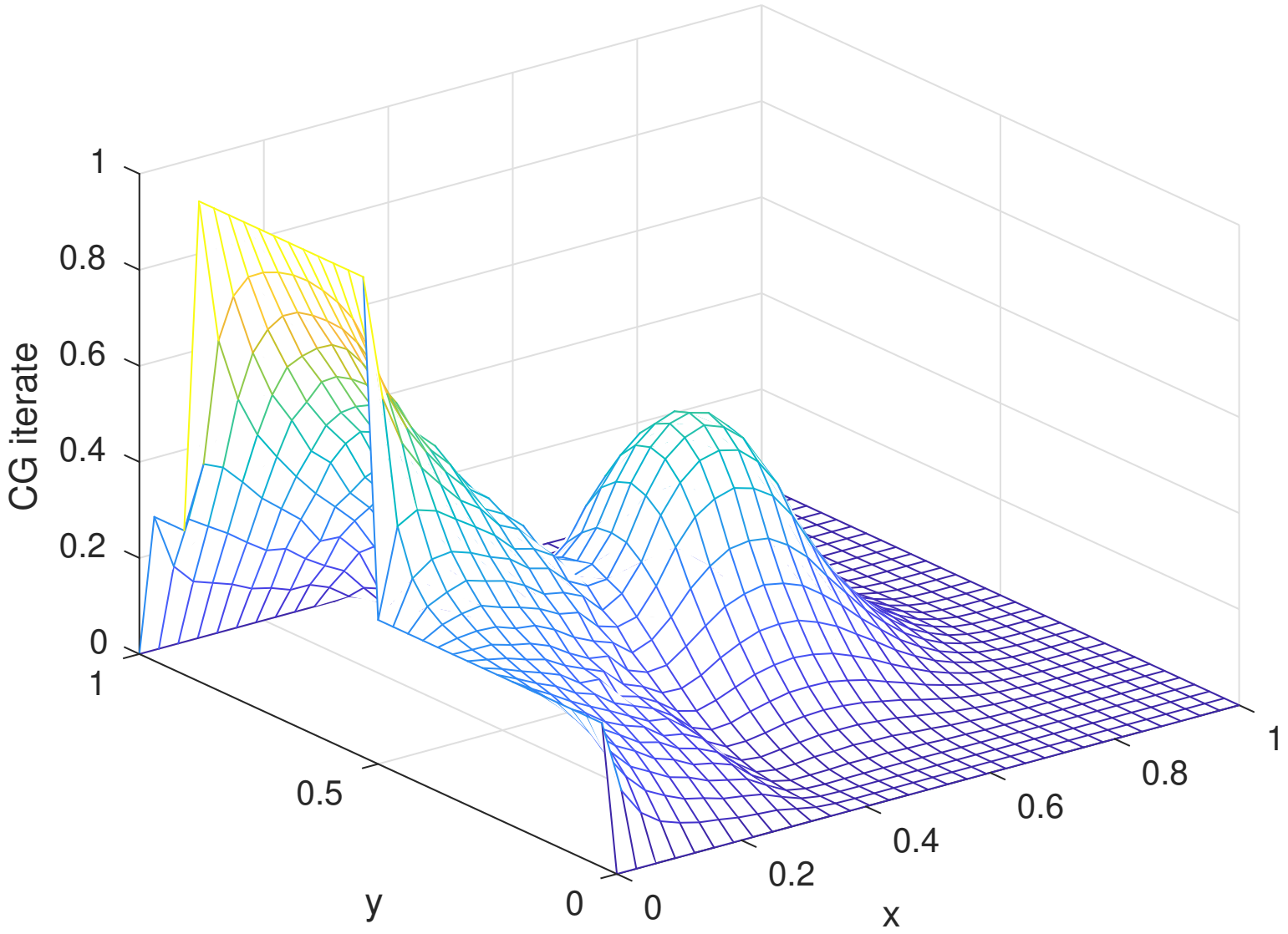
CG iteration 12



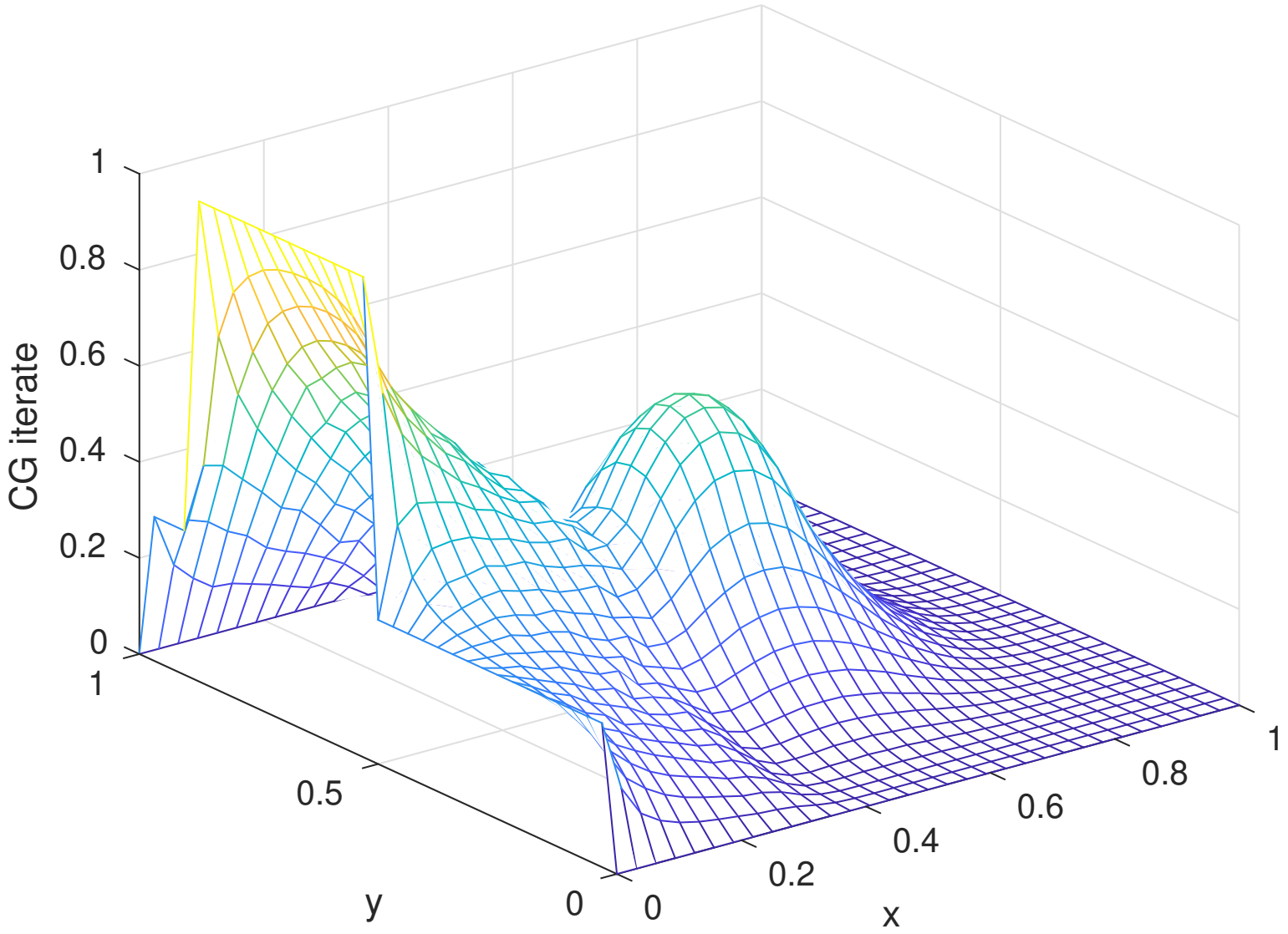
CG iteration 13



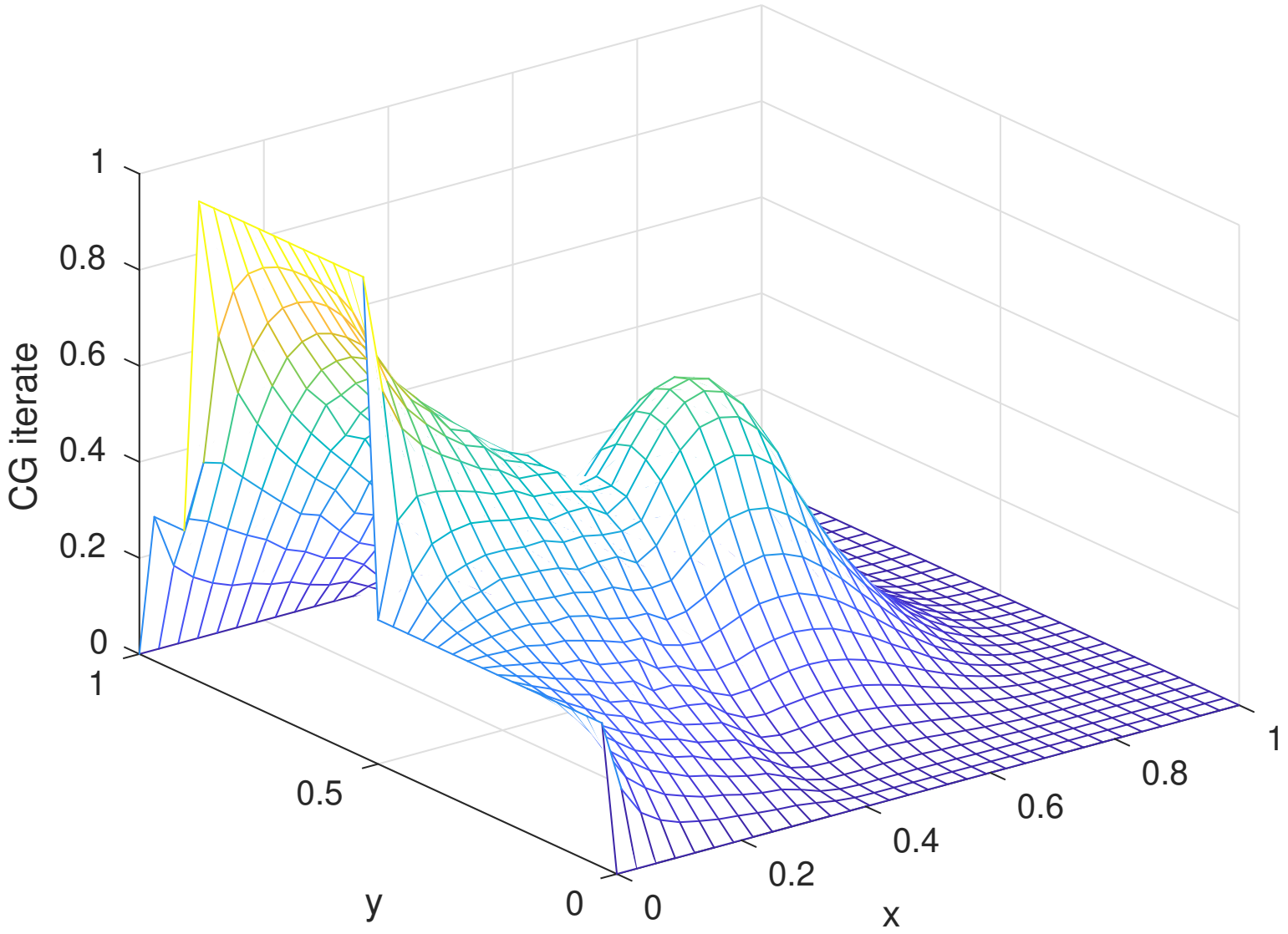
CG iteration 14



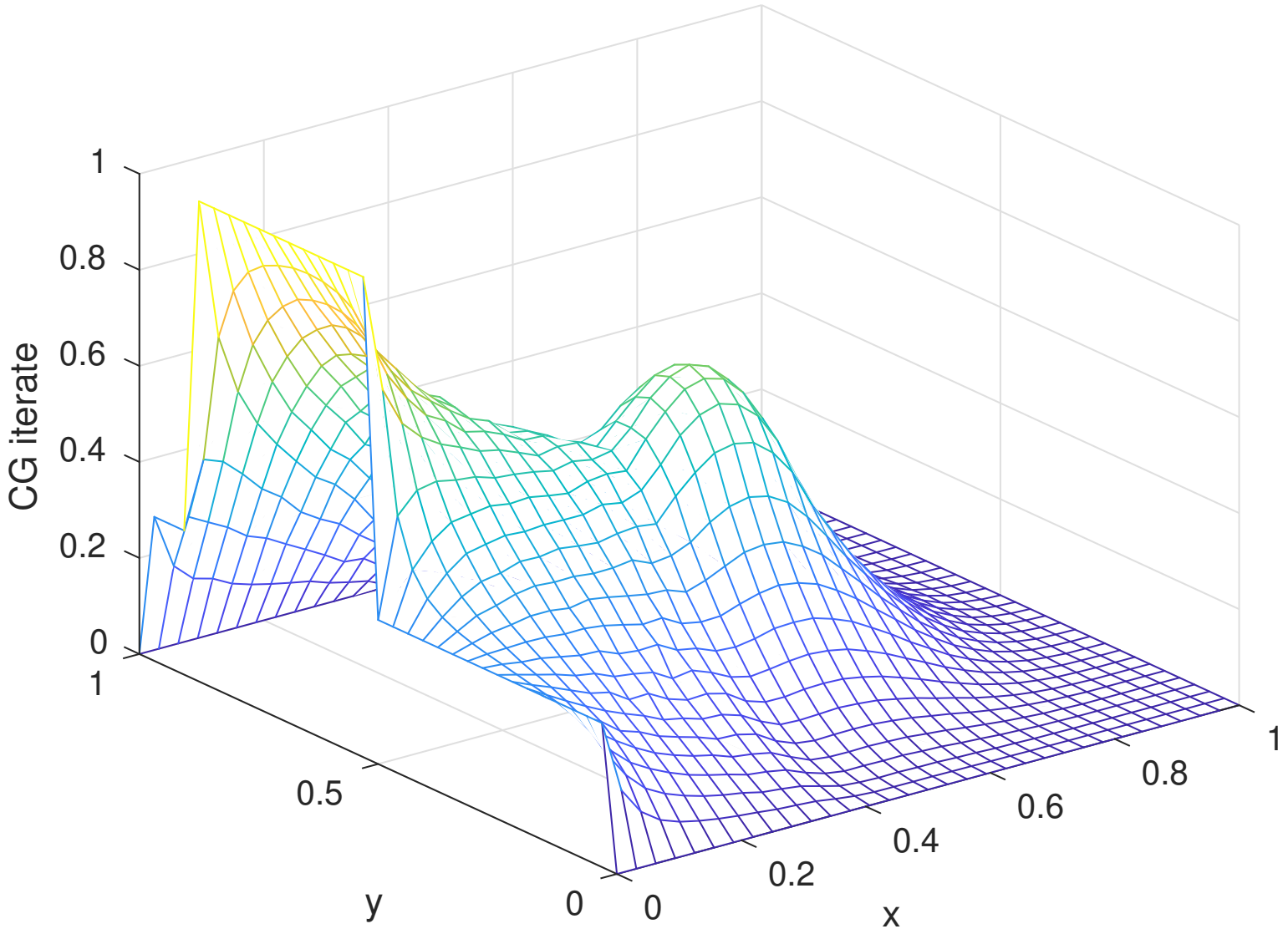
CG iteration 15



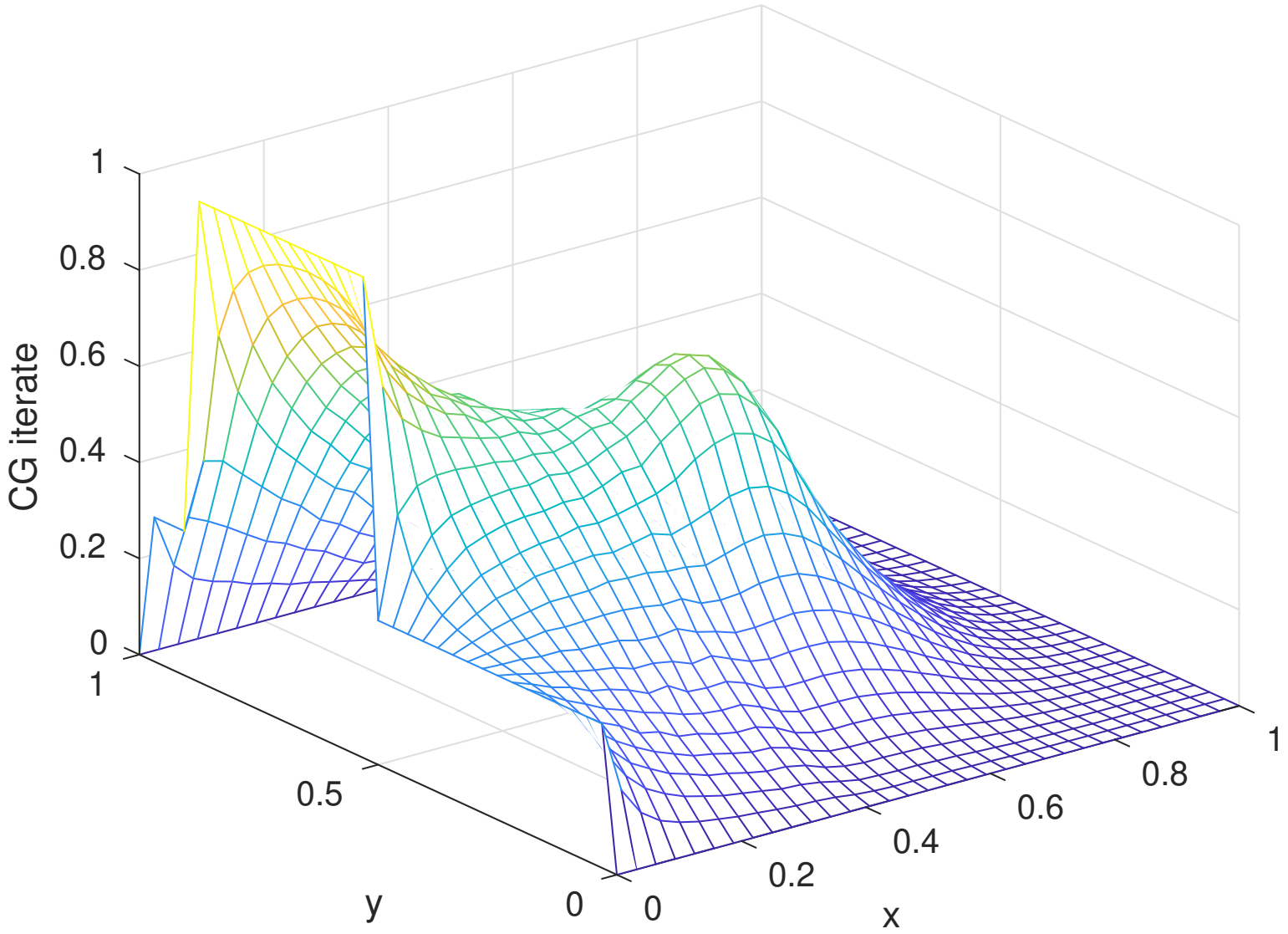
CG iteration 16



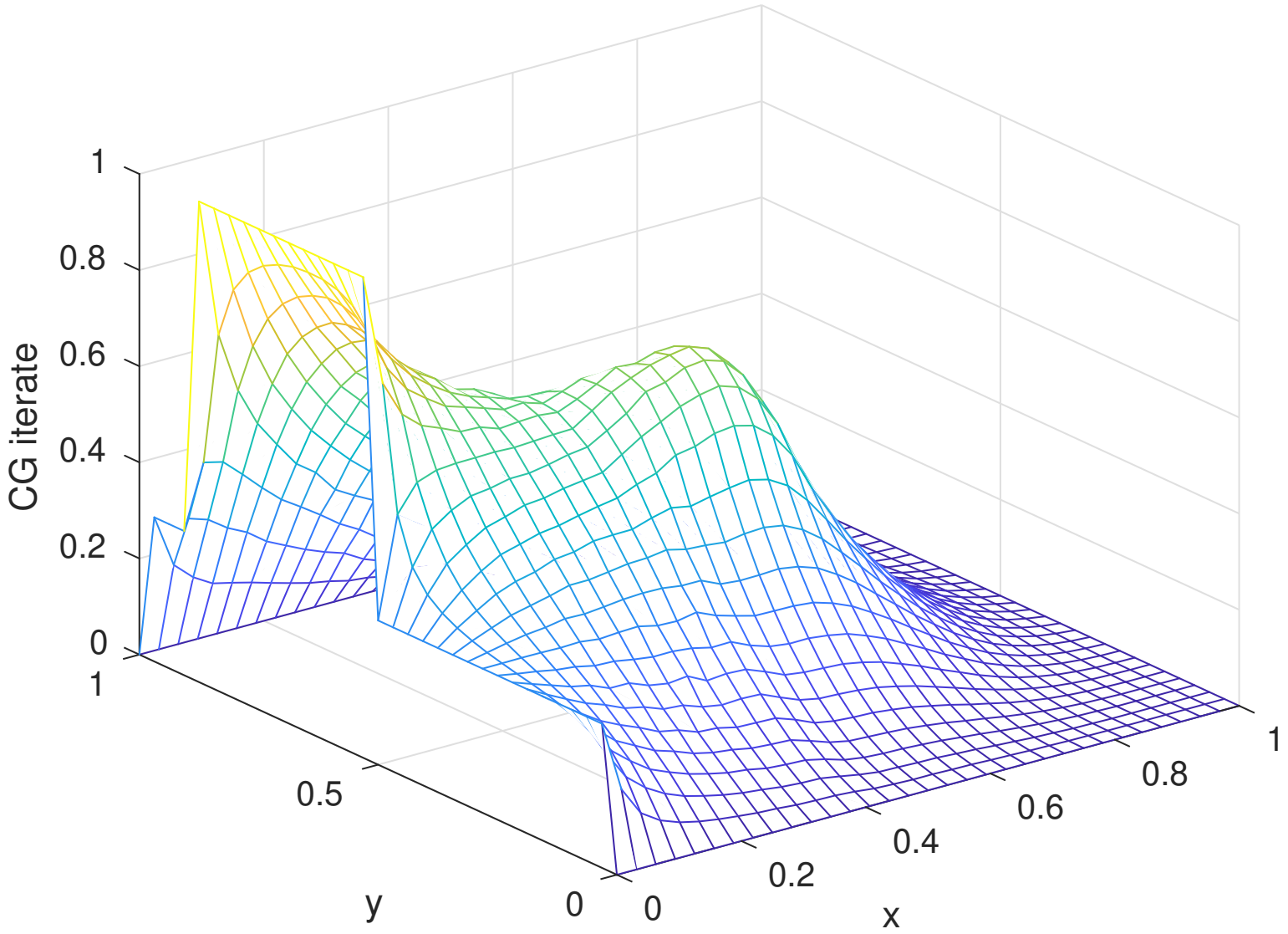
CG iteration 17



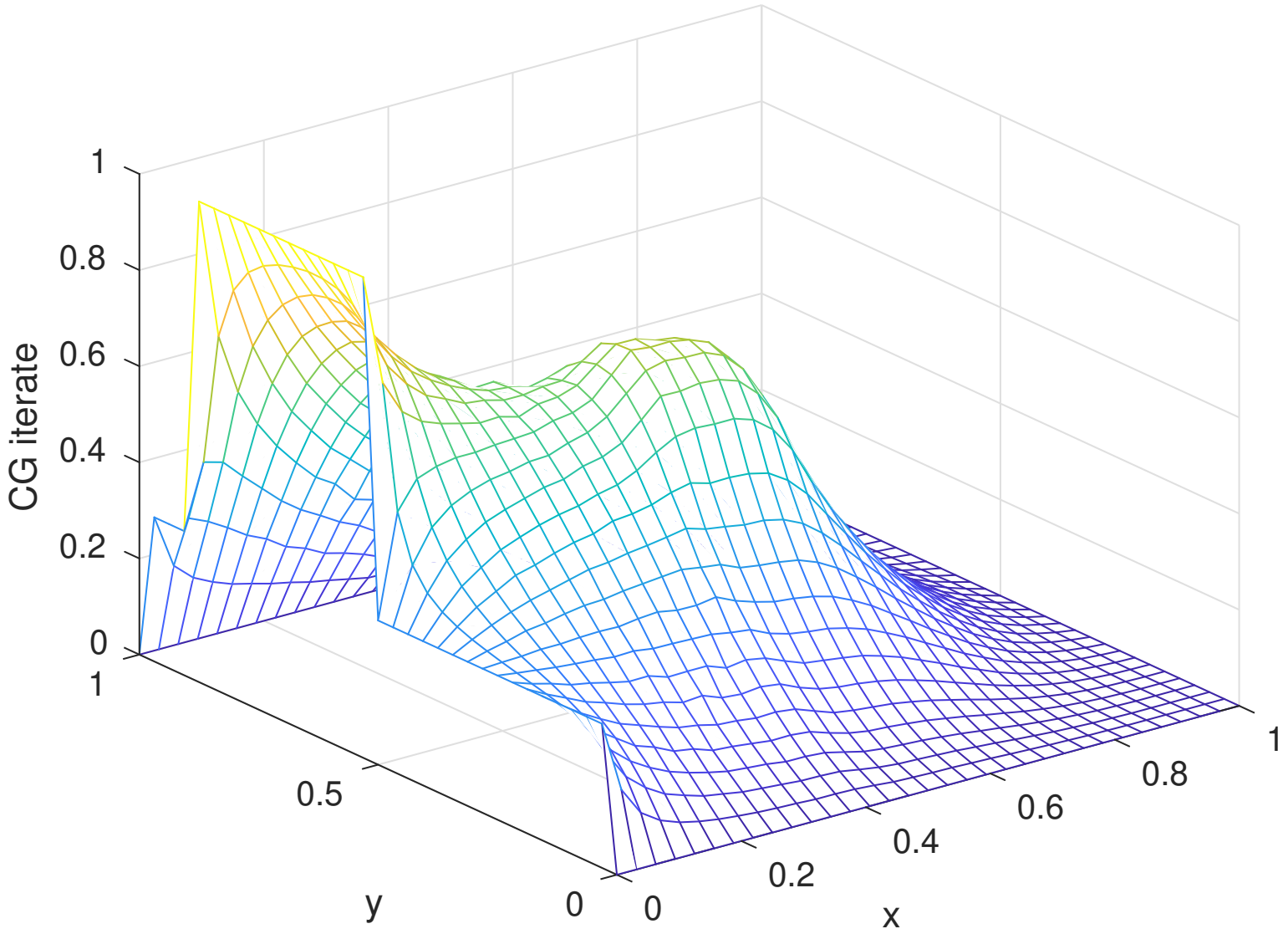
CG iteration 18



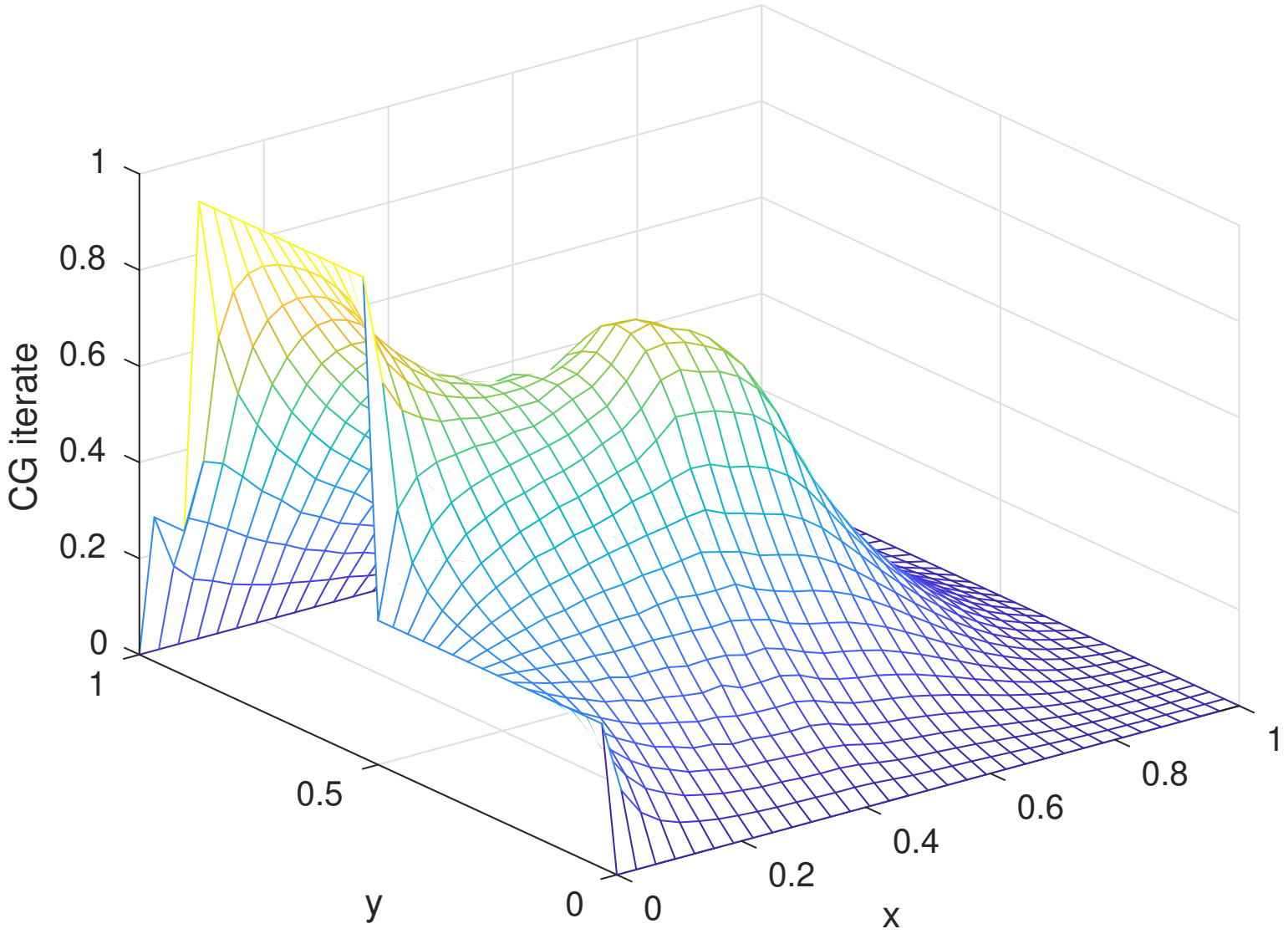
CG iteration 19



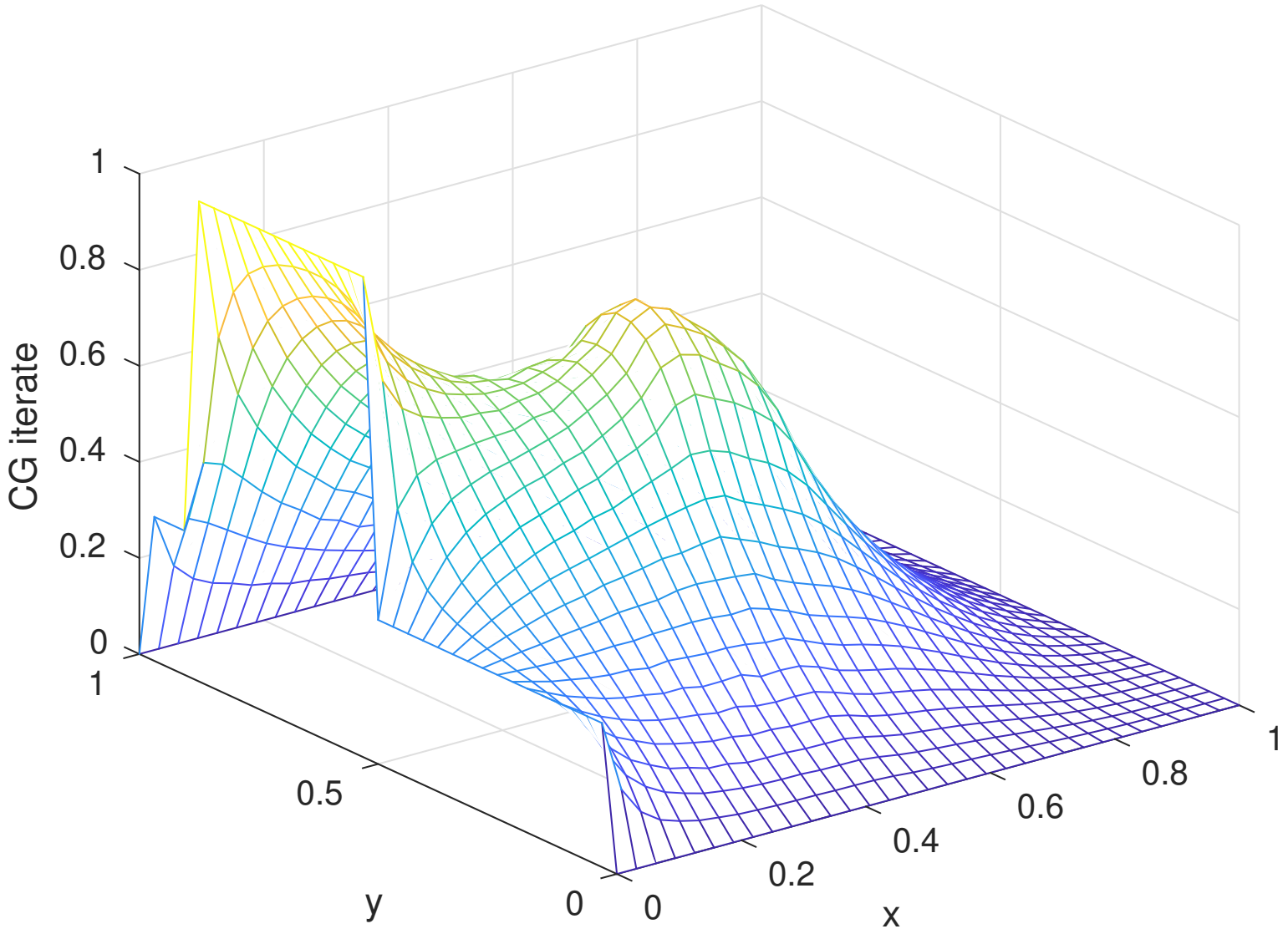
CG iteration 20



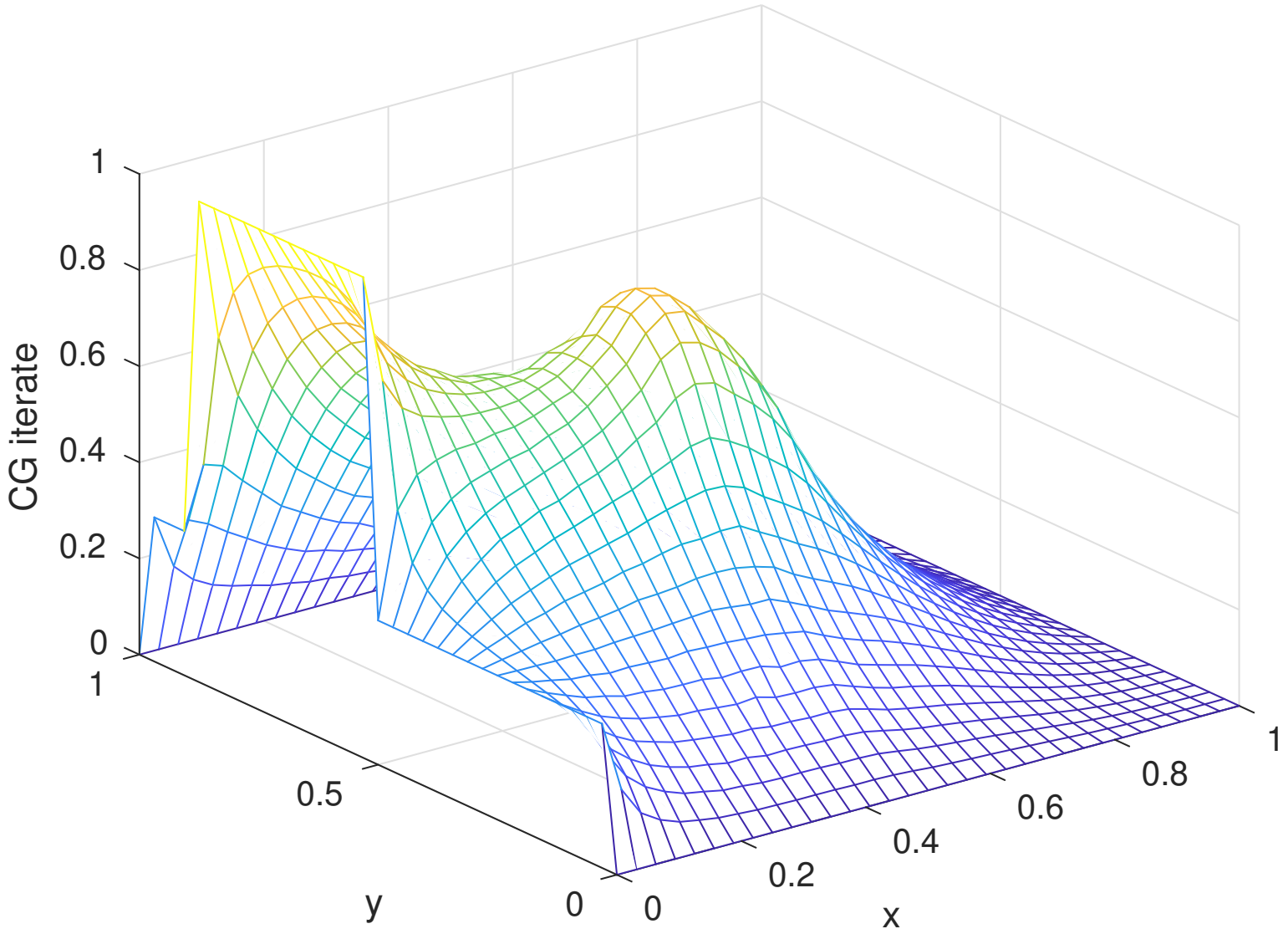
CG iteration 21



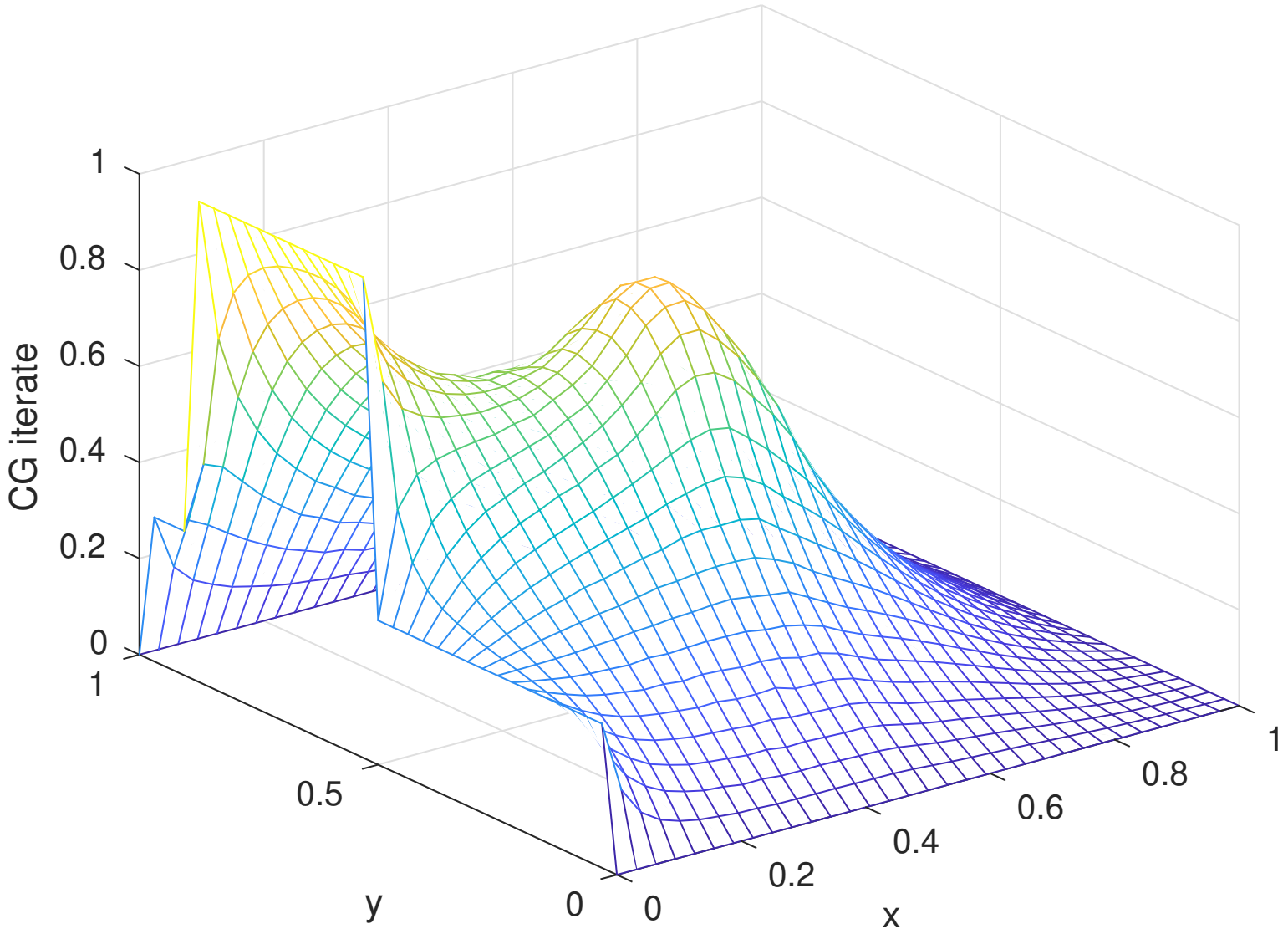
CG iteration 22



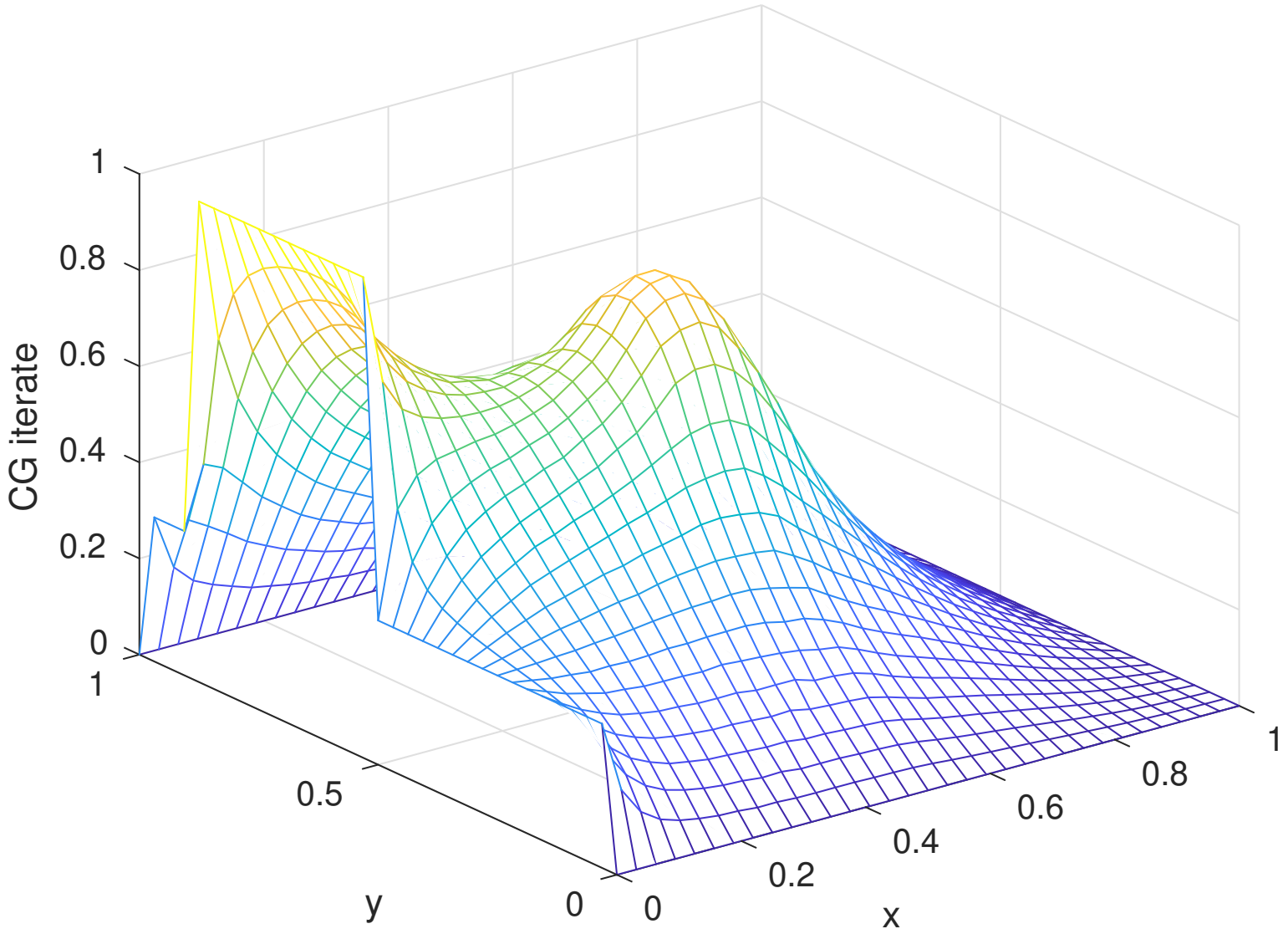
CG iteration 23



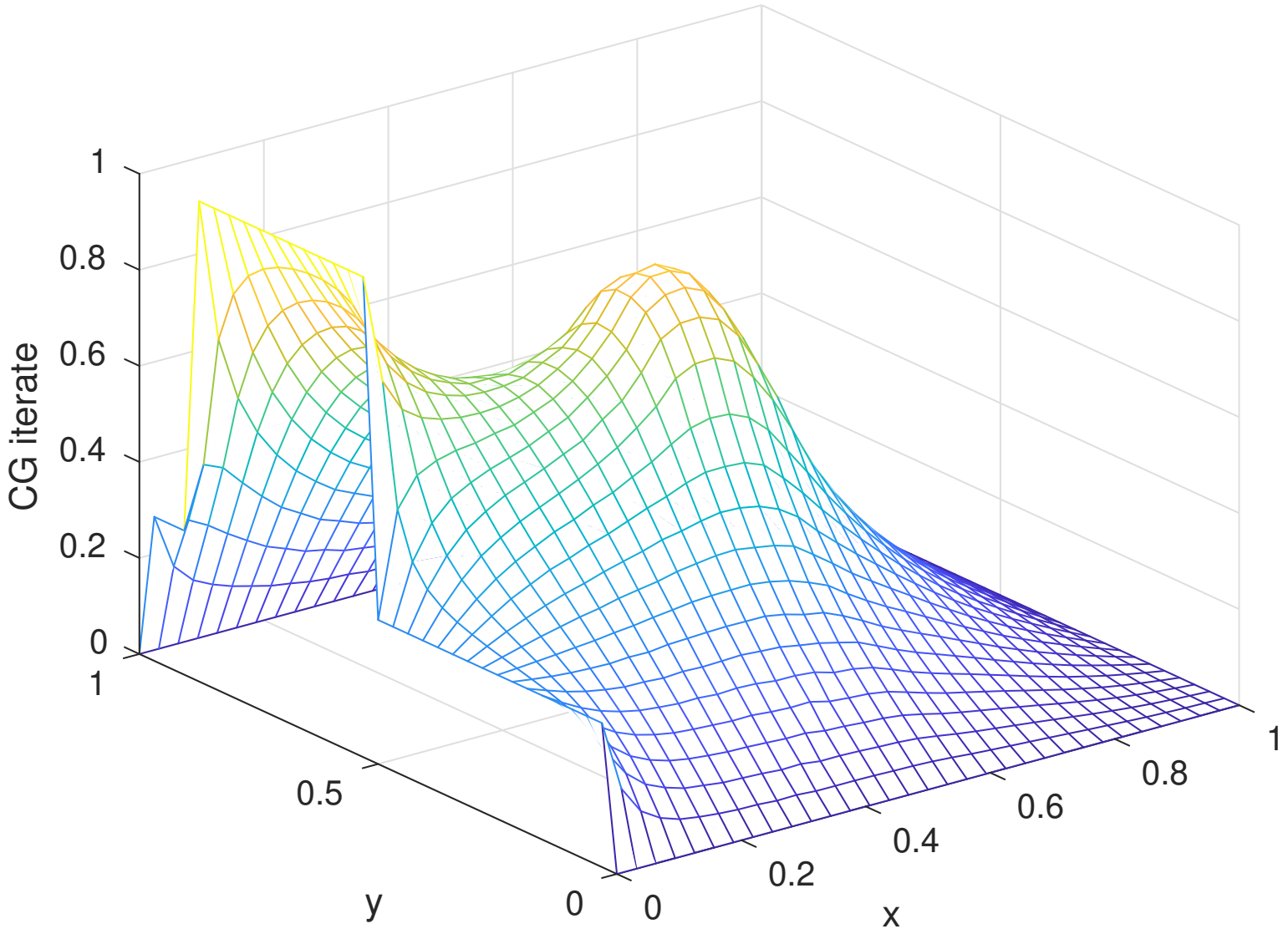
CG iteration 24



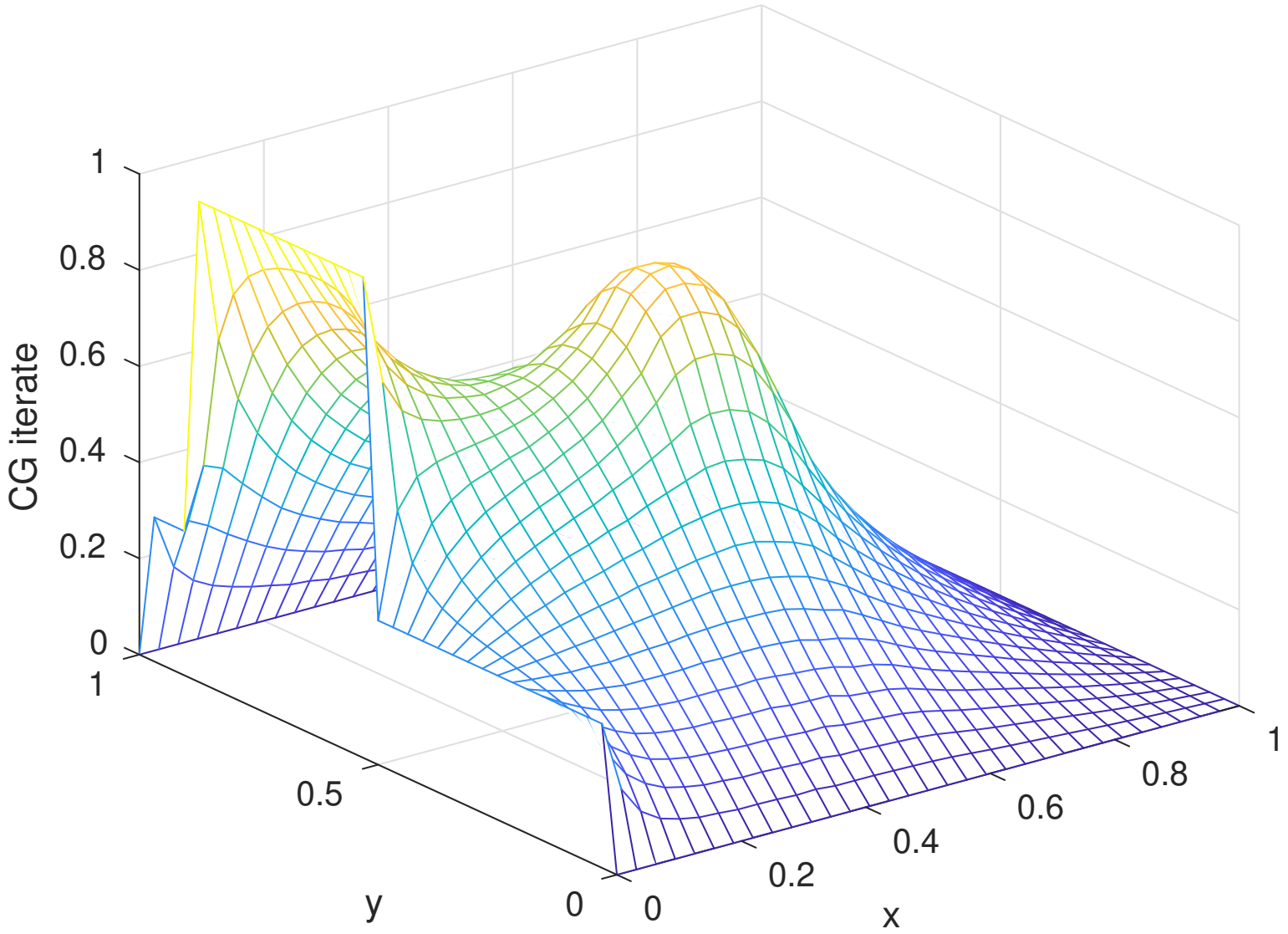
CG iteration 25



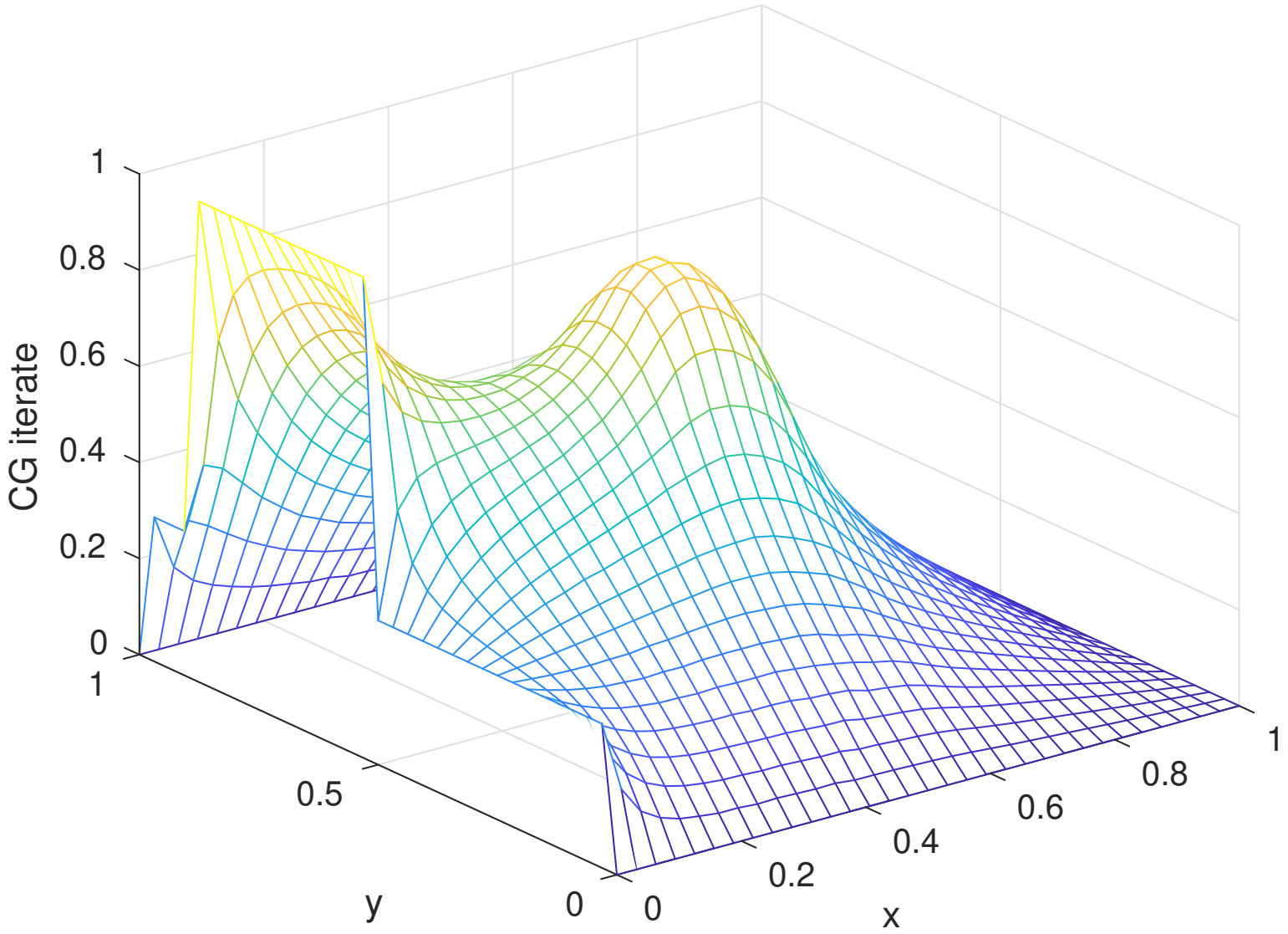
CG iteration 26



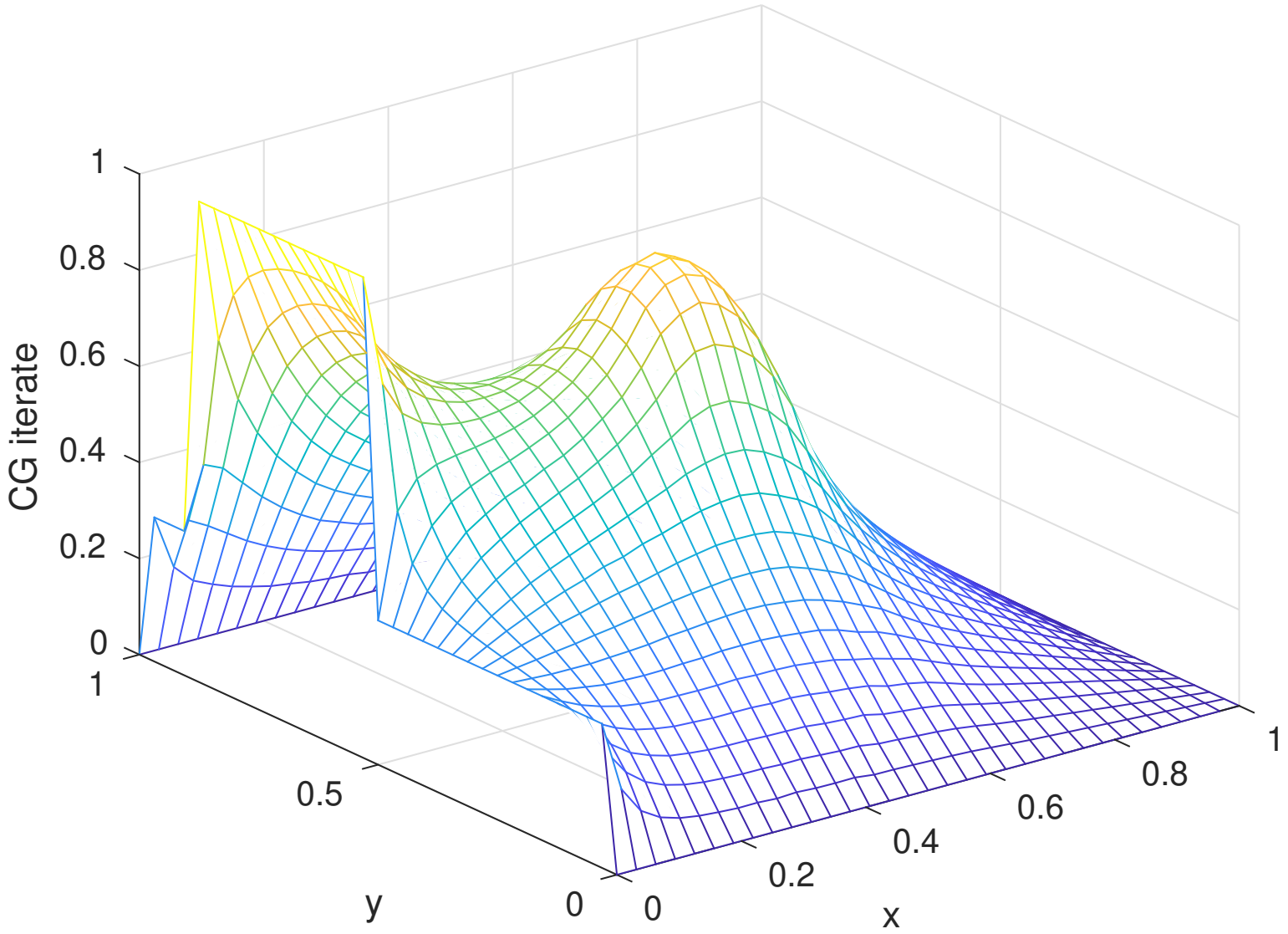
CG iteration 27



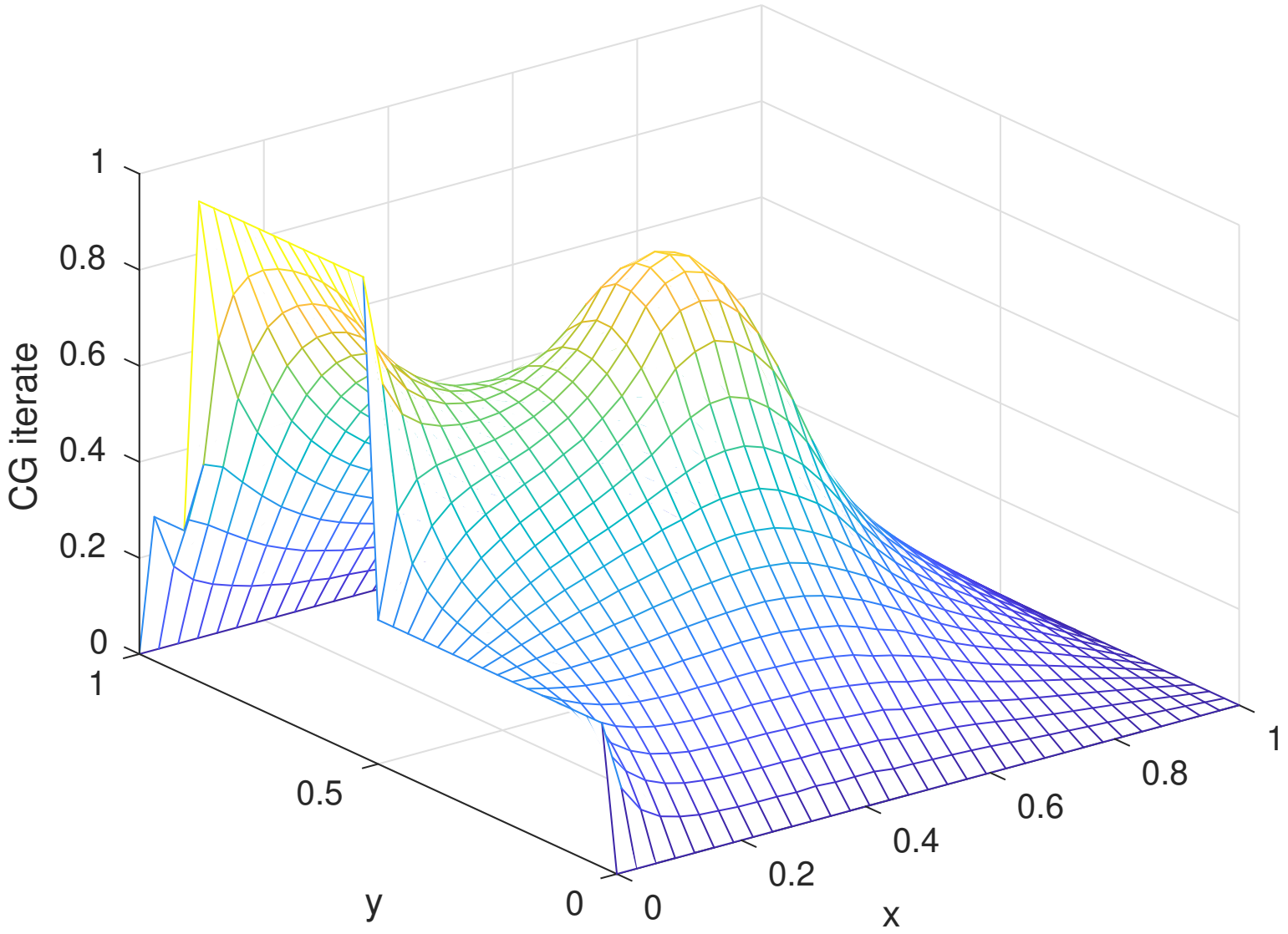
CG iteration 28



CG iteration 29



CG iteration 30



Conjugate Gradients in Modern Notation

To solve approximately $A\boldsymbol{u} = \boldsymbol{f}$, A spd, CG finds at step n in the Krylov space

$$\mathcal{K}_n(A, \boldsymbol{f}) := \{\boldsymbol{f}, A\boldsymbol{f}, \ldots, A^{n-1}\boldsymbol{f}\}$$

an approximate solution $\boldsymbol{u}_n$ which satisfies

$$||\boldsymbol{u} - \boldsymbol{u}_n||_A \longrightarrow \min .$$

In general, Krylov methods search in the Krylov space $\mathcal{K}_n$ an approximation minimizing either the error or the residual in some norm.

Krylov 1931: On the numerical solution of the equation by which, in technical matters, frequencies of small oscillations of material systems are determined

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Generalizations of Conjugate Gradients

- Minimum residual methods (MR): find $\boldsymbol{u}^n \in \boldsymbol{u}_0 + \mathcal{K}_n(A, \boldsymbol{r}^0)$ such that

$$||\boldsymbol{f} - A\boldsymbol{u}^n||_2 \longrightarrow \min .$$

  - MINRES (Paige, Saunders 1975)
  - GMRES (Saad, Schultz 1986)
  - QMR (Freund, Nachtigal 1991)

- Methods based on orthogonalization (OR): find $\boldsymbol{u}^n \in \boldsymbol{u}_0 + \mathcal{K}_n(A, \boldsymbol{r}^0)$ such that

$$\boldsymbol{f} - A\boldsymbol{u}^n \perp \mathcal{K}_n(A, \boldsymbol{r}^0).$$

  - SymmLQ (Paige, Saunders 1975)
  - FOM (Saad 1981)
  - BiCGstab (Van Der Vorst 1992)

For normal matrices, all these methods converge well, if the spectrum of the matrix A is clustered around 1.

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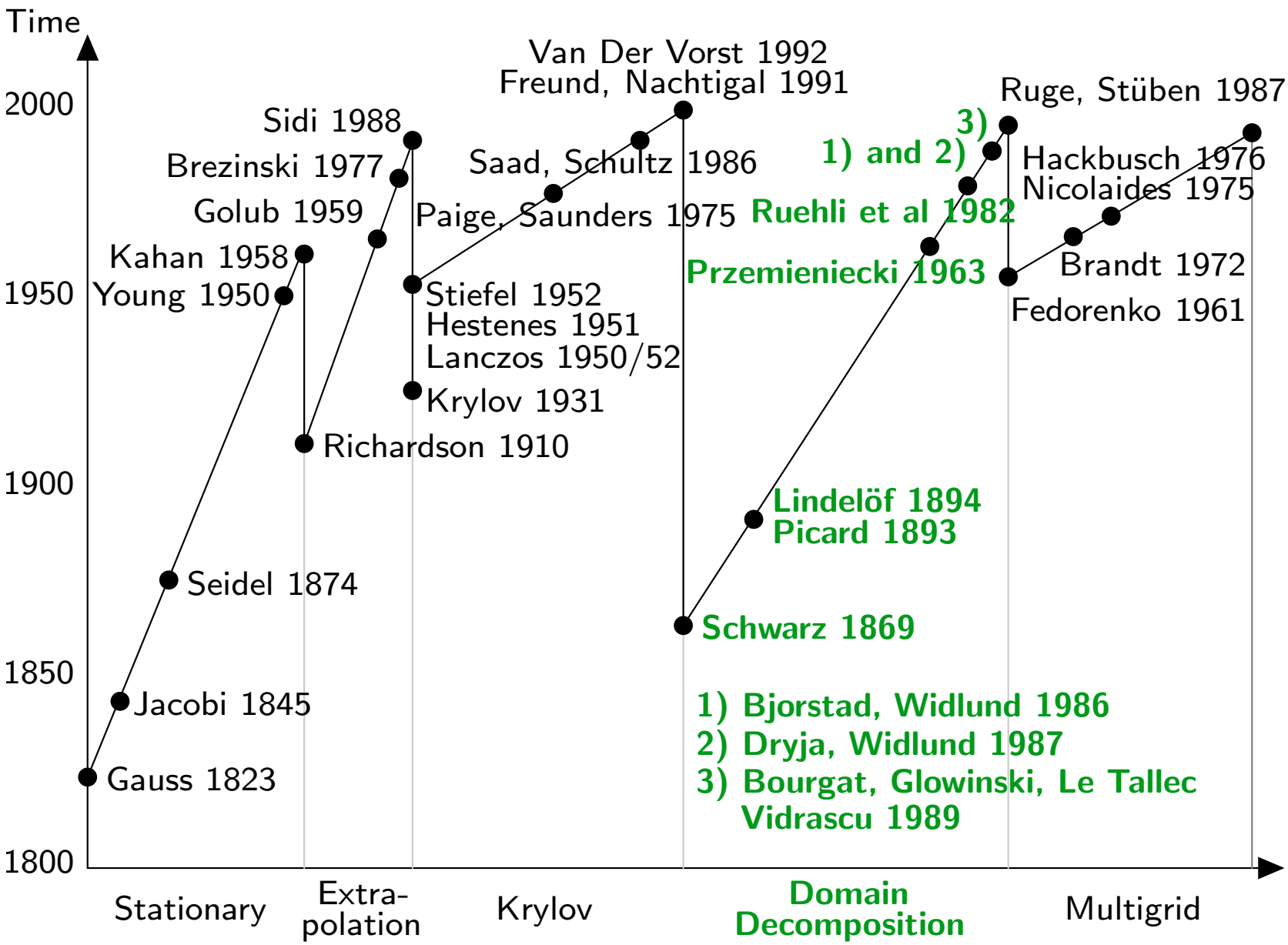

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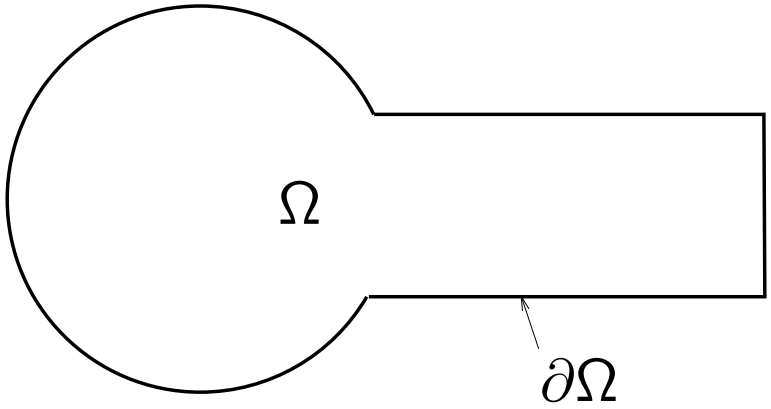
H.A. Schwarz (1869) Über einen Grenzübergang durch alternierendes Verfahren

$$\Delta u = 0 \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega$$



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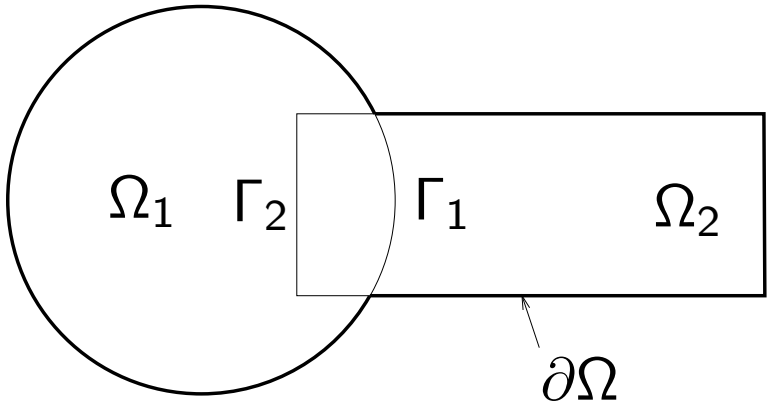

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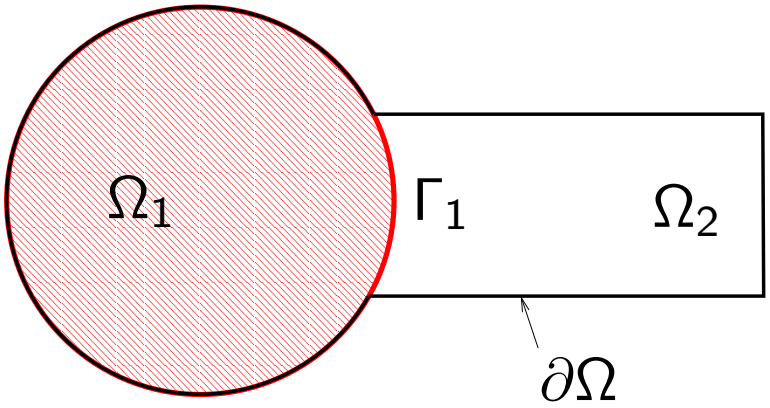

$$\begin{aligned} \Delta u_1^1 &= 0 && \text{in } \Omega_1 \\ u_1^1 &= g && \text{on } \partial\Omega \cap \overline{\Omega}_1 \\ u_1^1 &= u_2^0 && \text{on } \Gamma_1 \end{aligned}$$

solve on the disk $u_2^0 = 0$

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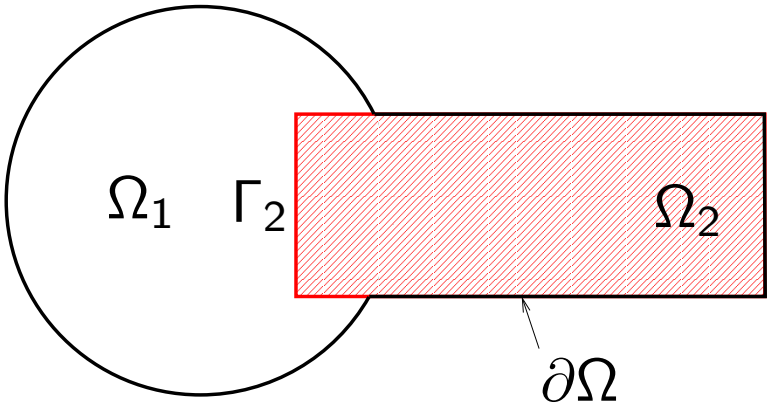

$$\begin{aligned} \Delta u_2^1 &= 0 && \text{in } \Omega_2 \\ u_2^1 &= g && \text{on } \partial\Omega \cap \overline{\Omega}_2 \\ u_2^1 &= u_1^1 && \text{on } \Gamma_2 \end{aligned}$$

solve on the rectangle

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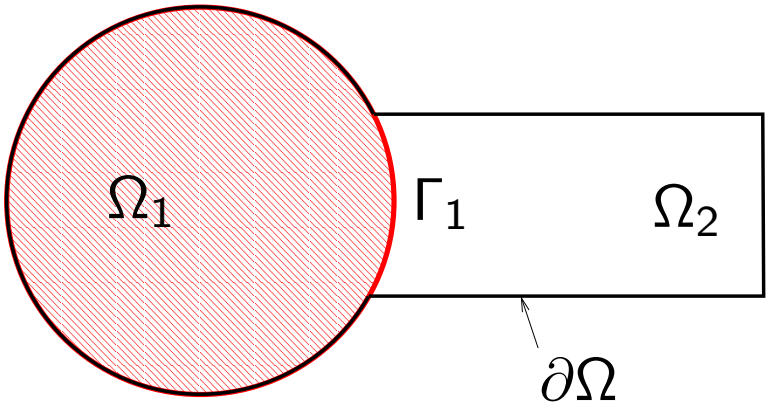

$$\begin{aligned} \Delta u_1^2 &= 0 && \text{in } \Omega_1 \\ u_1^2 &= g && \text{on } \partial\Omega \cap \overline{\Omega}_1 \\ u_1^2 &= u_2^1 && \text{on } \Gamma_1 \end{aligned}$$

solve on the disk

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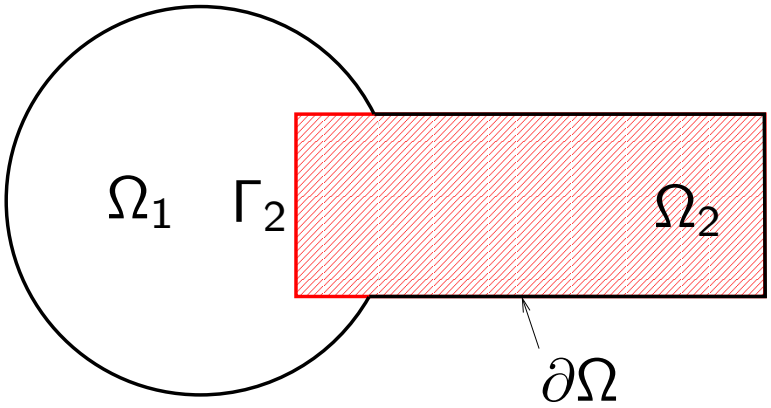

$$\begin{aligned} \Delta u_2^2 &= 0 \quad \text{in } \Omega_2 \\ u_2^2 &= g \quad \text{on } \partial\Omega \cap \overline{\Omega}_2 \\ u_2^2 &= u_1^2 \quad \text{on } \Gamma_2 \end{aligned}$$

solve on the rectangle

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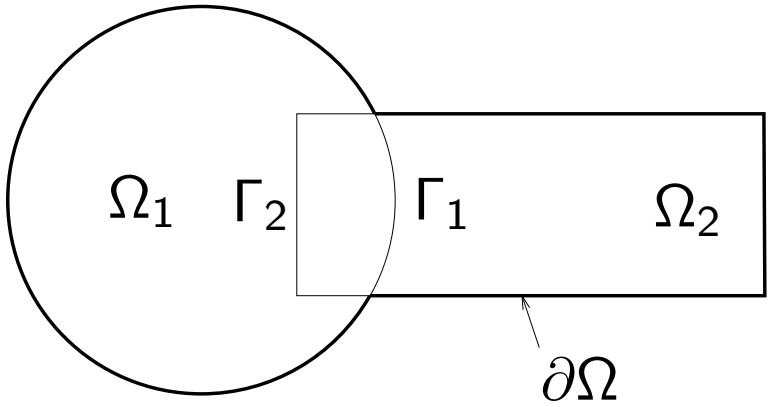

$$\begin{aligned} \Delta u_1^n &= 0 && \text{in } \Omega_1 \\ u_1^n &= g && \text{on } \partial\Omega \cap \overline{\Omega}_1 \\ u_1^n &= u_2^{n-1} && \text{on } \Gamma_1 \end{aligned}$$

solve on the disk

$$\begin{aligned} \Delta u_2^n &= 0 && \text{in } \Omega_2 \\ u_2^n &= g && \text{on } \partial\Omega \cap \overline{\Omega}_2 \\ u_2^n &= u_1^n && \text{on } \Gamma_2 \end{aligned}$$

solve on the rectangle

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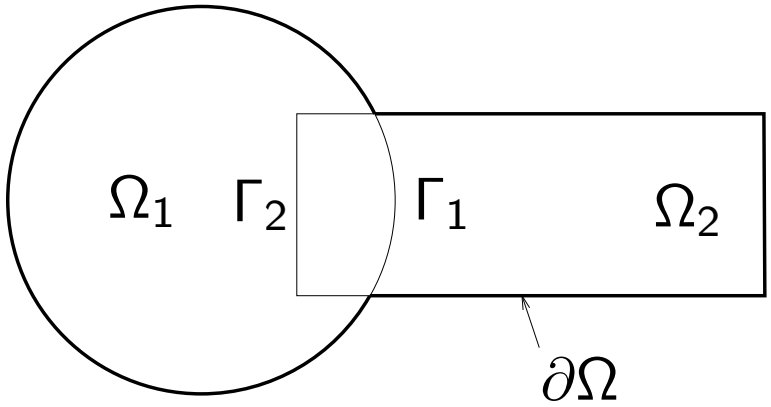
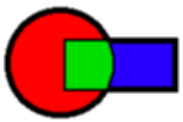


$$\begin{aligned} \Delta u_1^n &= 0 && \text{in } \Omega_1 \\ u_1^n &= g && \text{on } \partial\Omega \cap \overline{\Omega}_1 \\ u_1^n &= u_2^{n-1} && \text{on } \Gamma_1 \end{aligned} \qquad \begin{aligned} \Delta u_2^n &= 0 && \text{in } \Omega_2 \\ u_2^n &= g && \text{on } \partial\Omega \cap \overline{\Omega}_2 \\ u_2^n &= u_1^n && \text{on } \Gamma_2 \end{aligned}$$

solve on the disk — solve on the rectangle



Schwarz proved convergence in 1869 using the maximum principle

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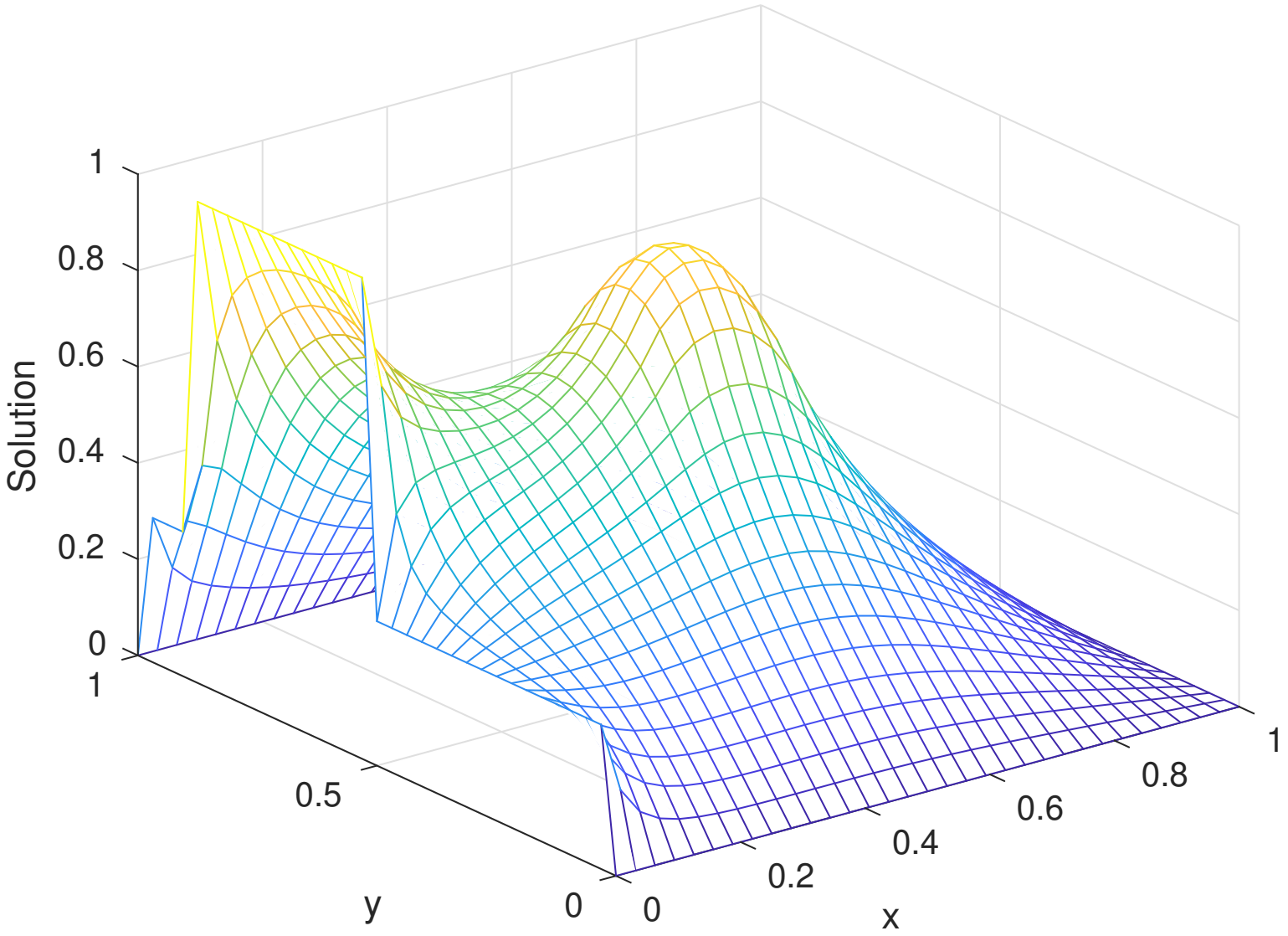
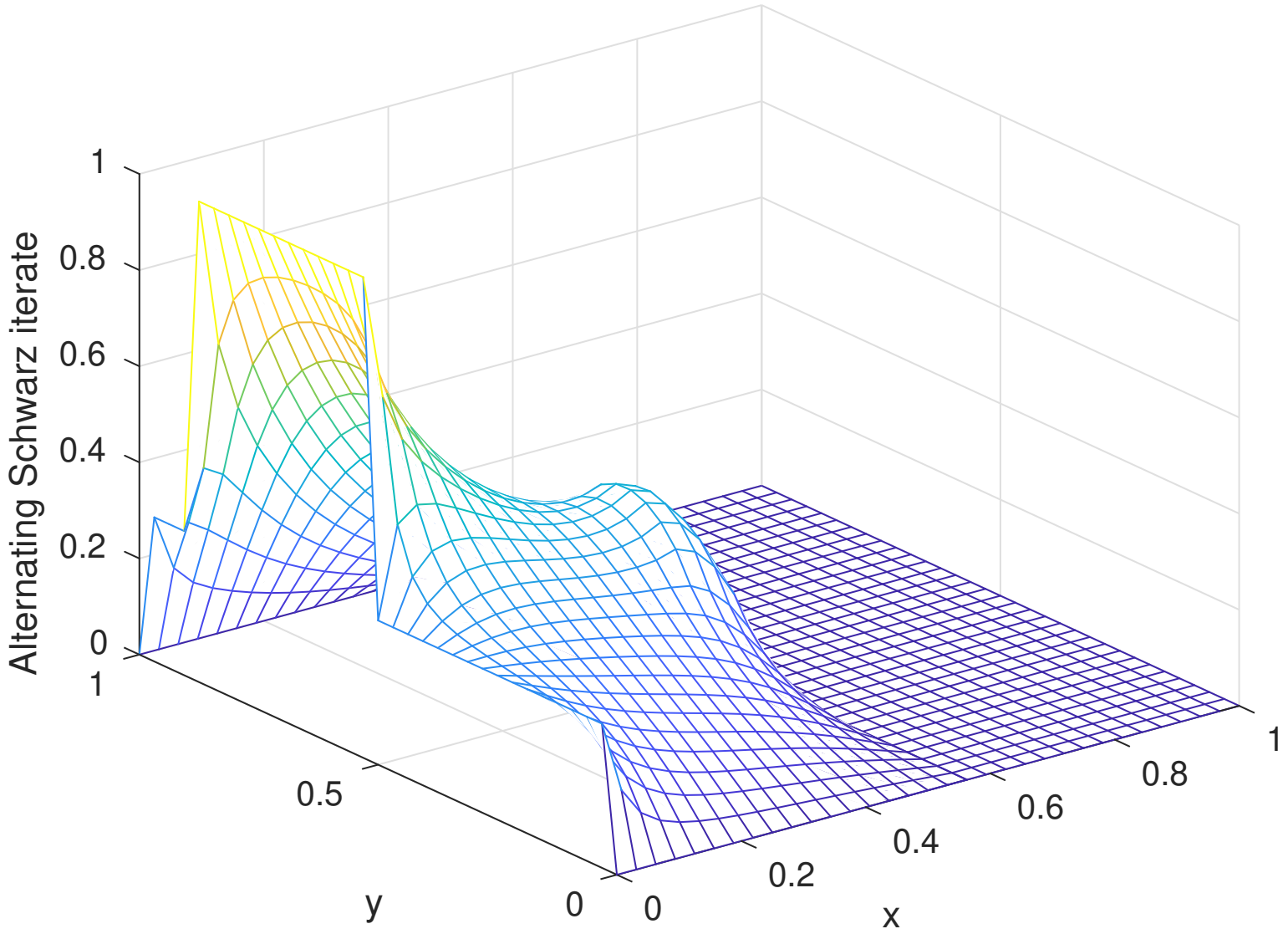
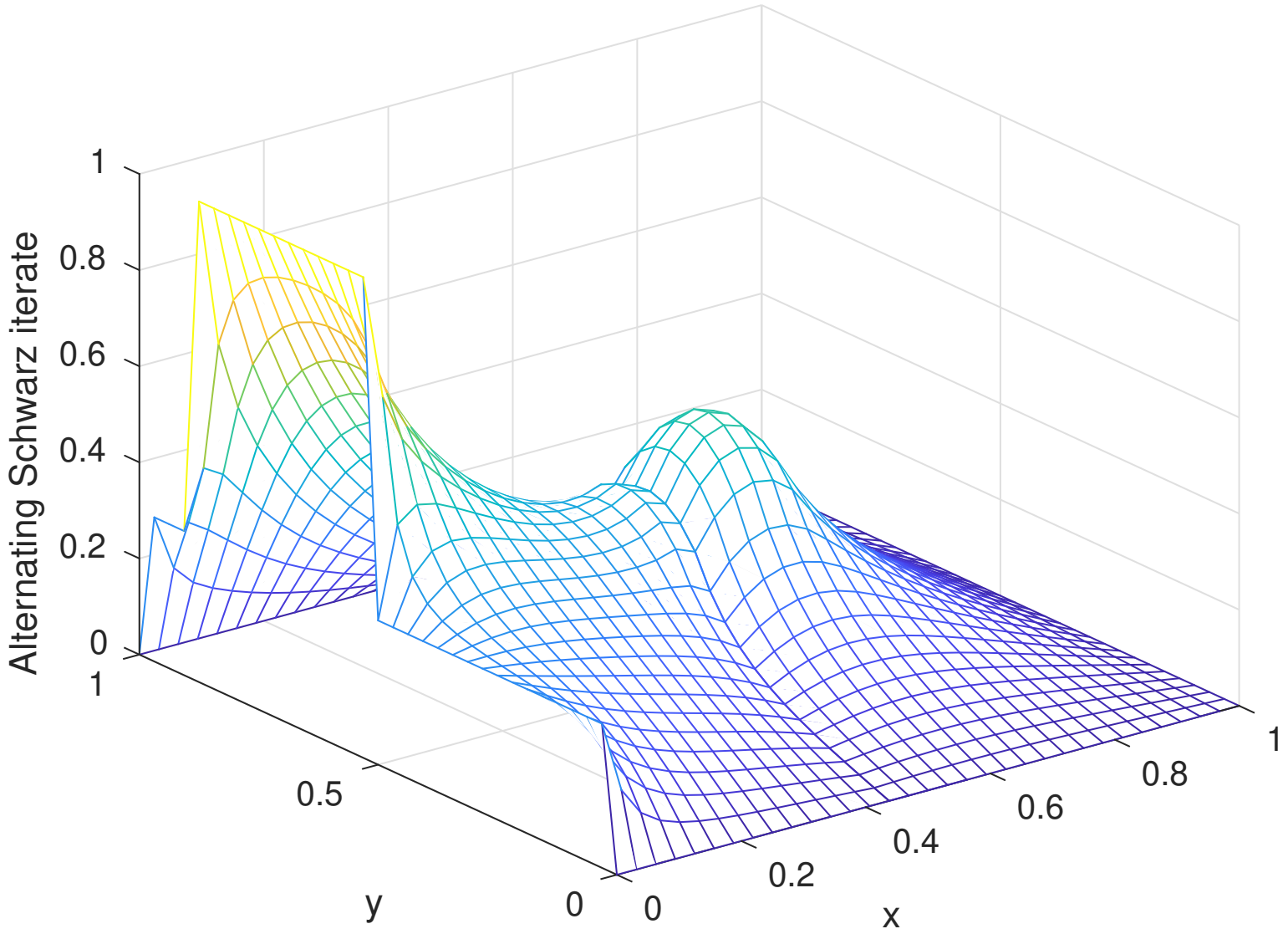
Example: temperature in a room



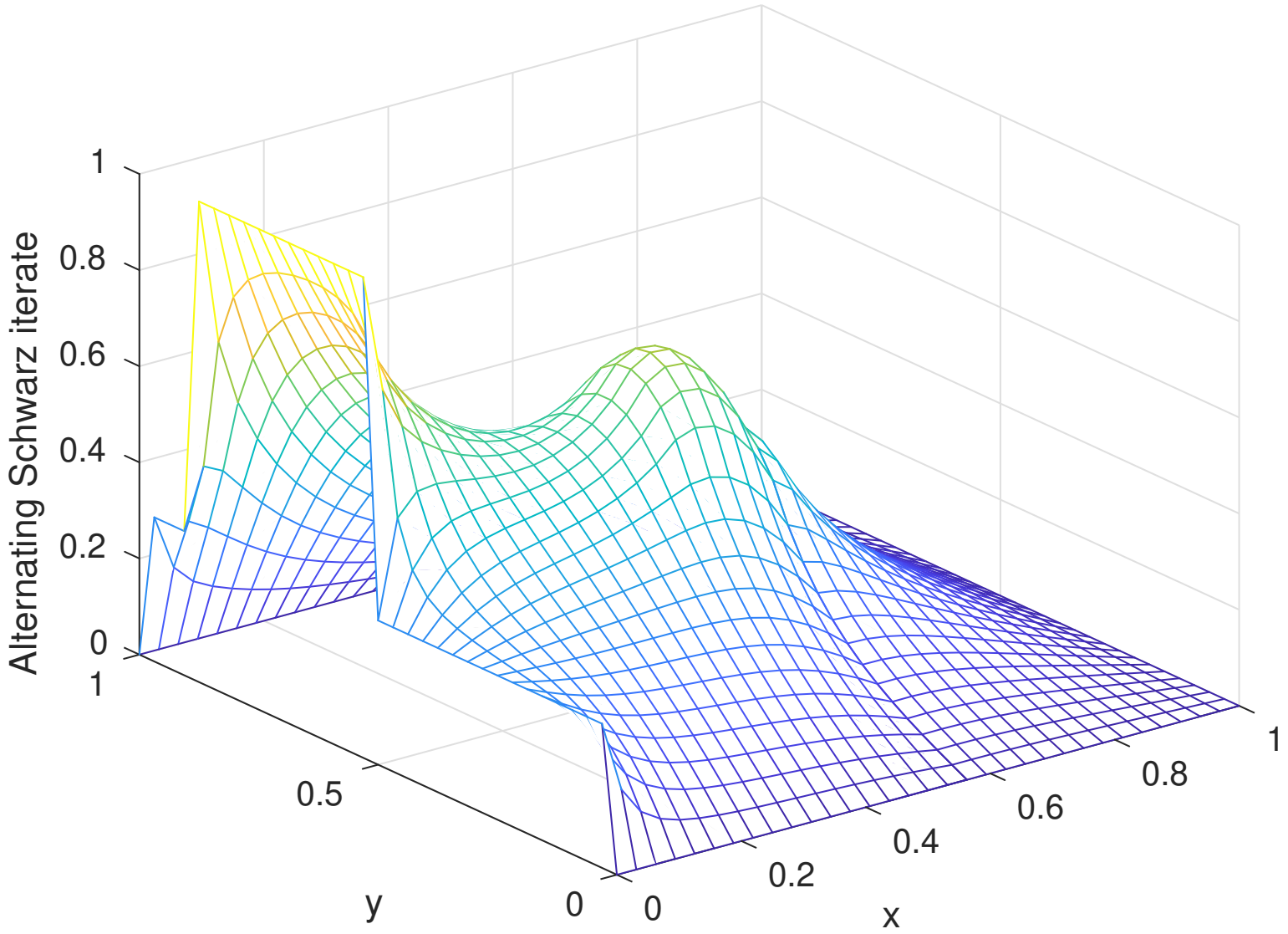
Alternating Schwarz iteration 1 on Ω_1



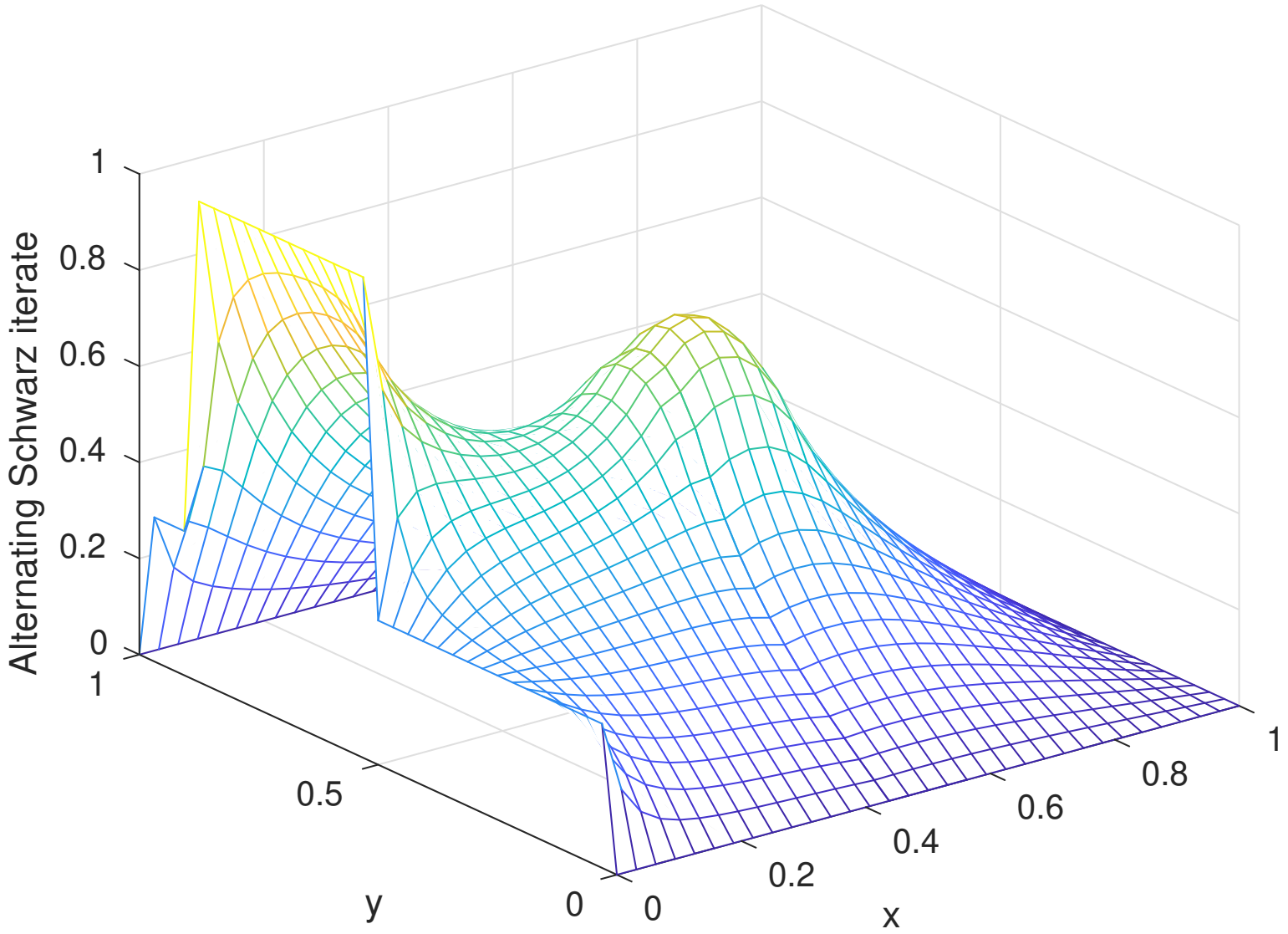
Alternating Schwarz iteration 1 on Ω_2



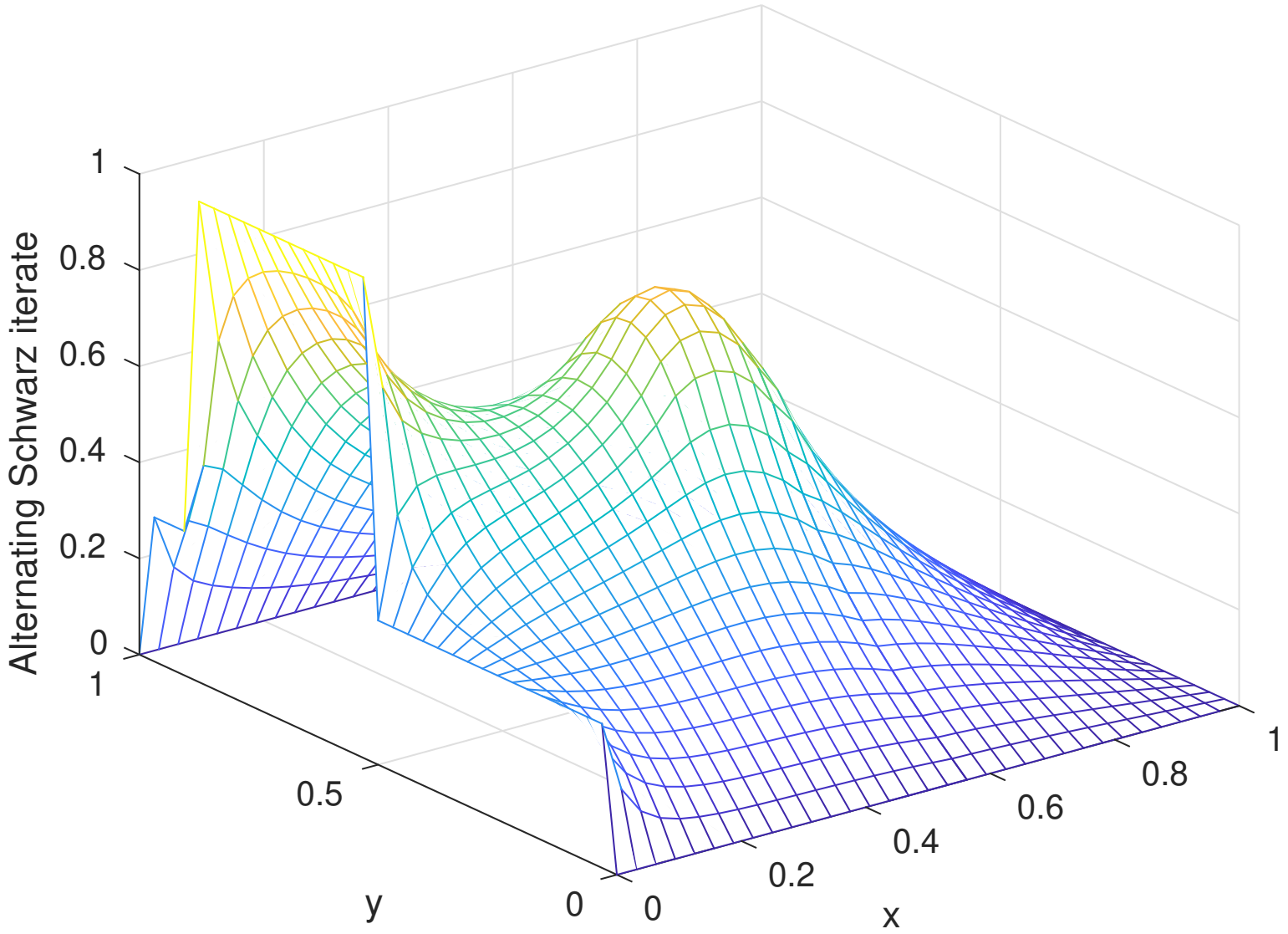
Alternating Schwarz iteration 2 on Ω_1



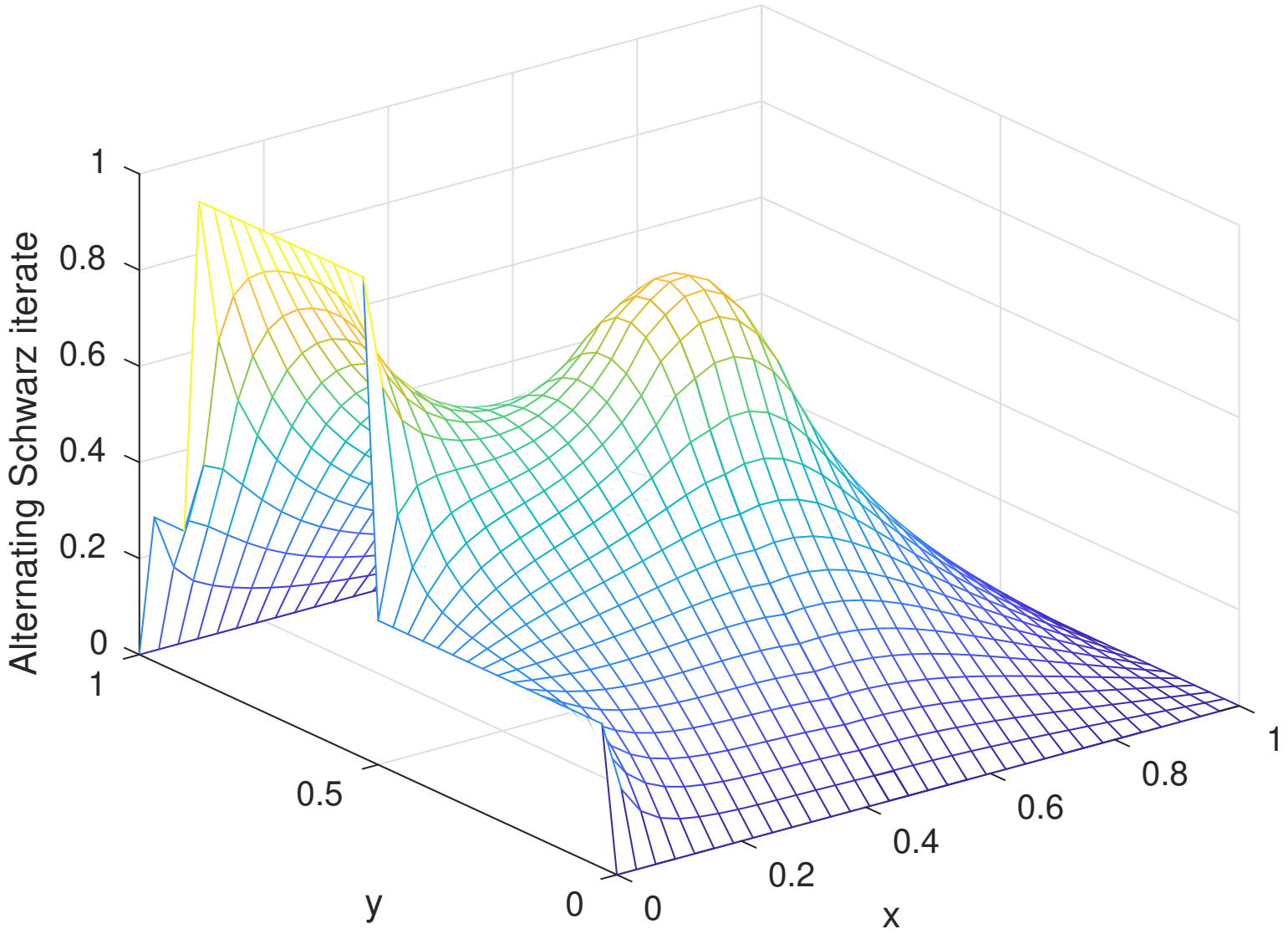
Alternating Schwarz iteration 2 on Ω_2



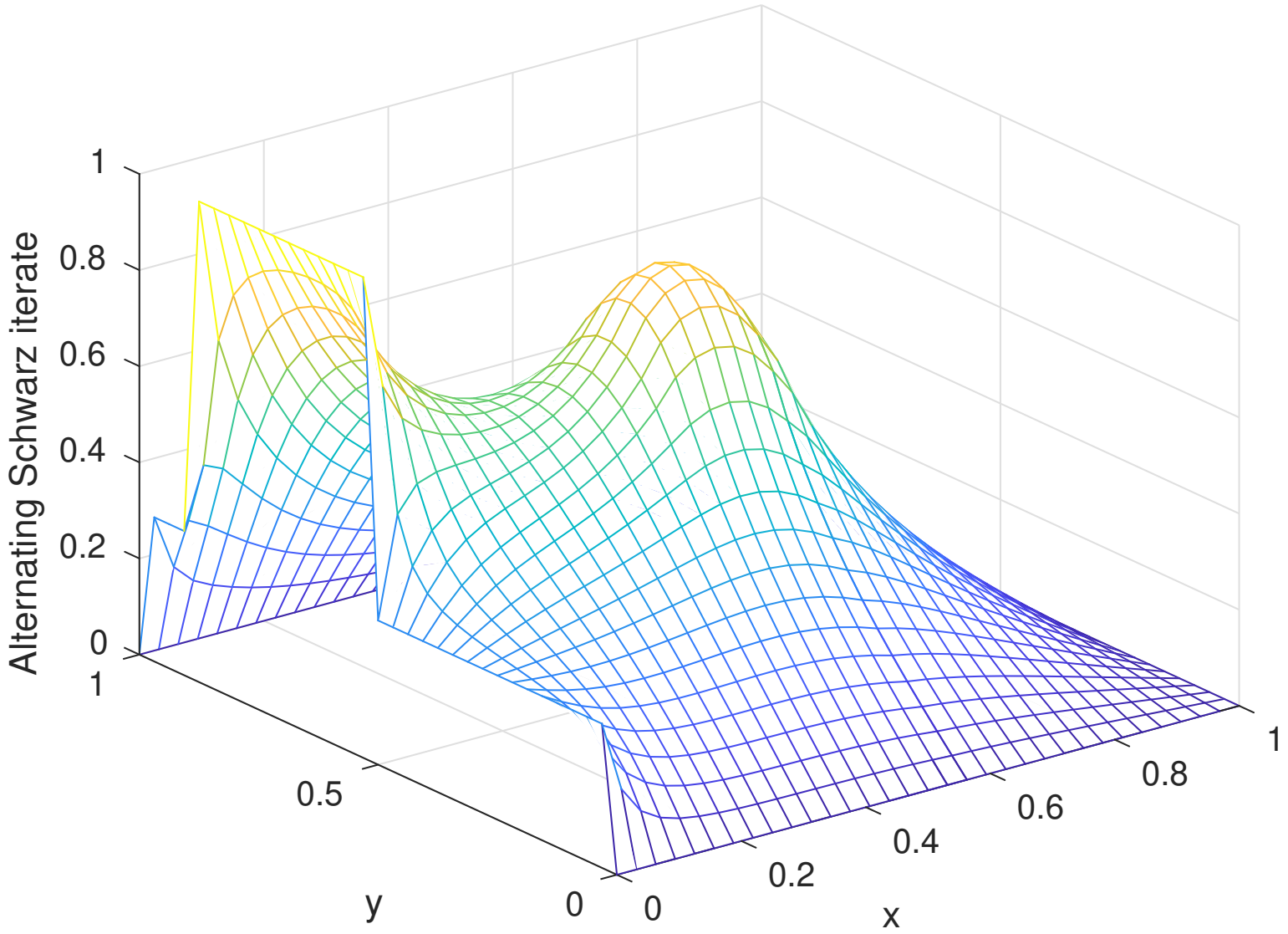
Alternating Schwarz iteration 3 on Ω_1



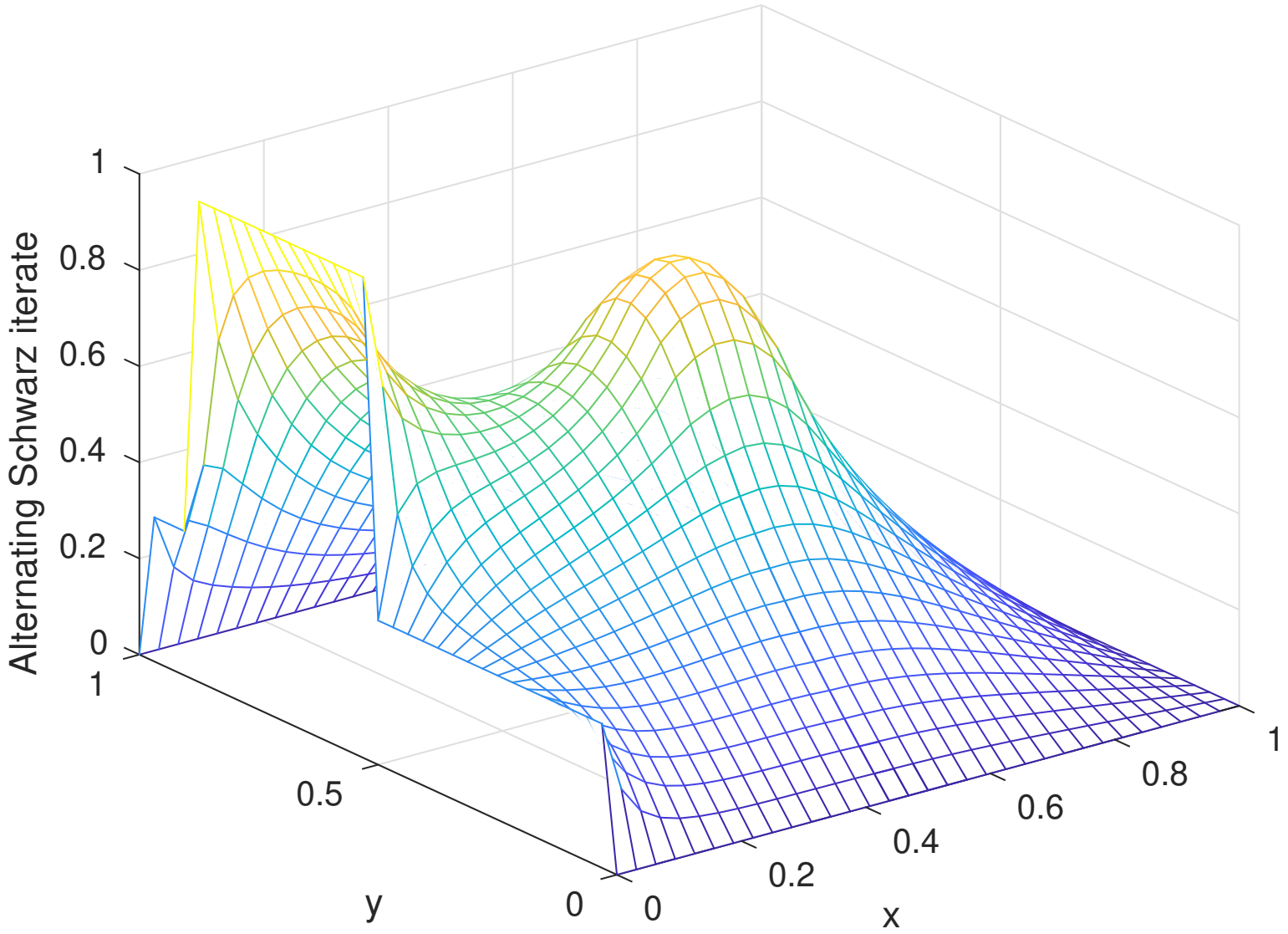
Alternating Schwarz iteration 3 on Ω_2



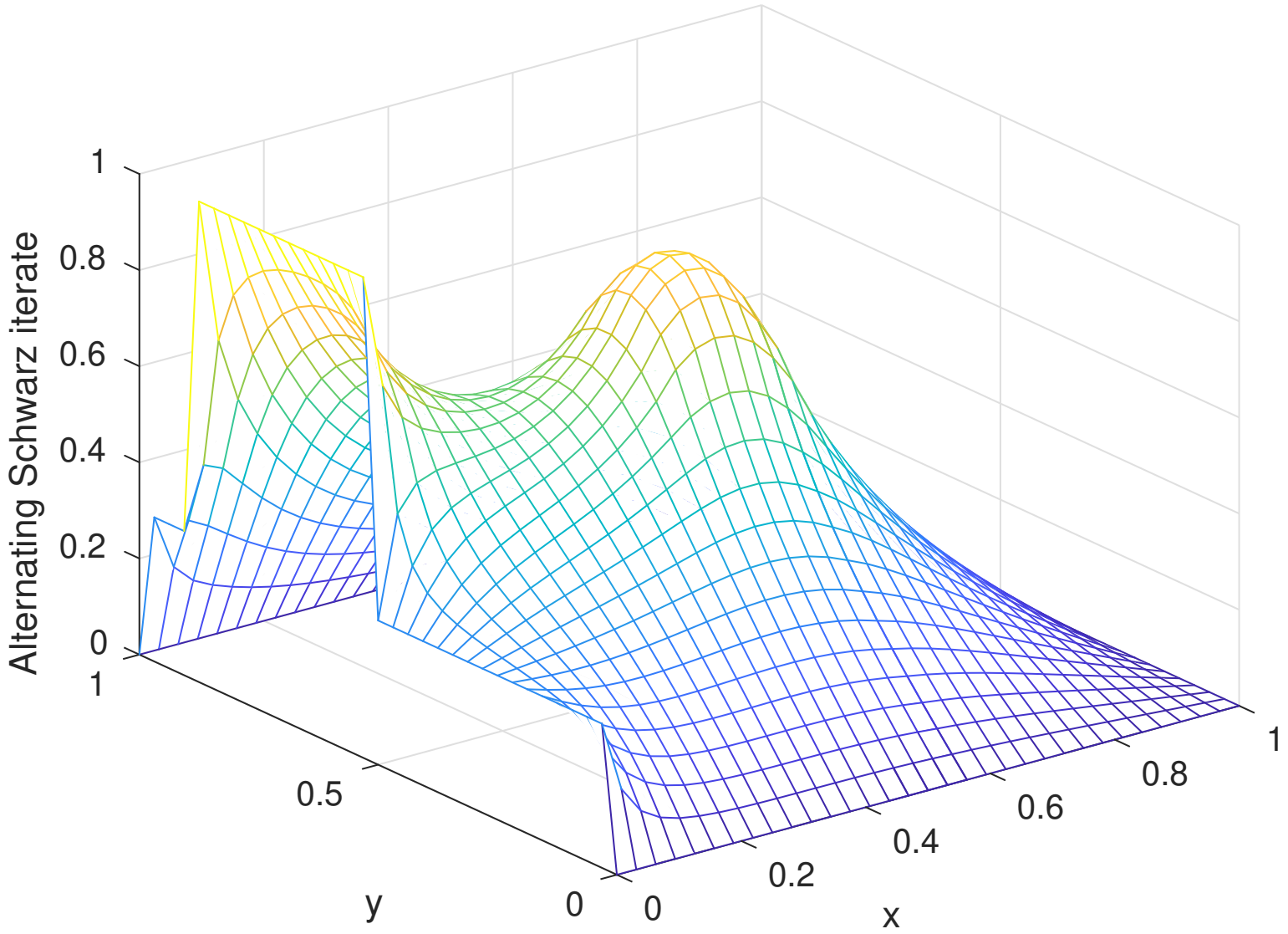
Alternating Schwarz iteration 4 on Ω_1



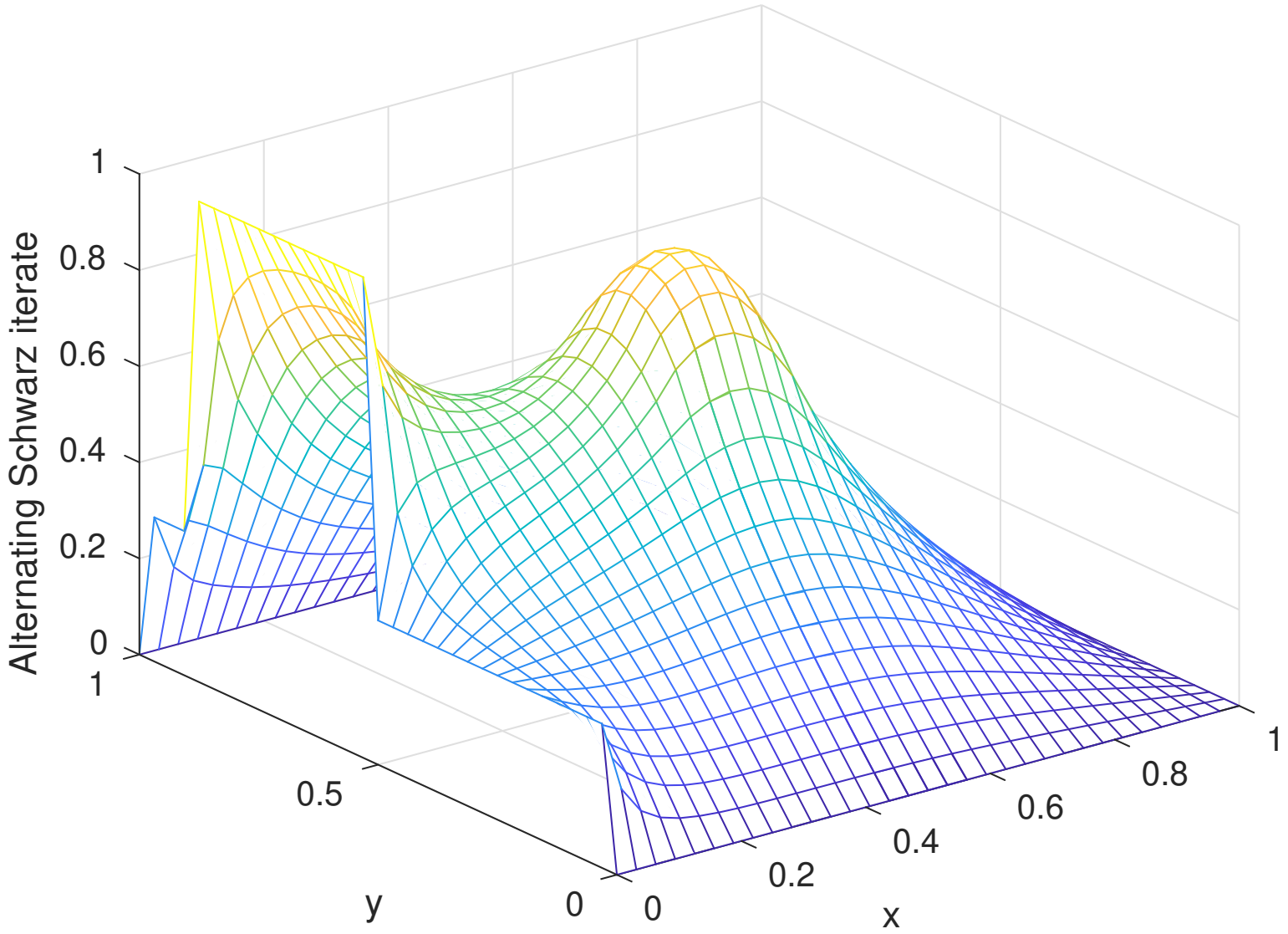
Alternating Schwarz iteration 4 on Ω_2



Alternating Schwarz iteration 5 on Ω_1



Alternating Schwarz iteration 5 on Ω_2



Further Domain Decomposition Methods

Przemieniecki (1963): Matrix structural analysis of substructures

Lelarasmee, Ruehli and Sangiovanni-Vincentelli (1982): The Waveform Relaxation Method for Time-Domain Analysis of Large Scale Integrated Circuits.

Bjørstad, Widlund (1986): Iterative methods for the solution of elliptic problems on regions partitioned into substructures (Dirichlet Neumann method)

Dryja and Widlund (1987): An Additive Variant of the Schwarz Alternating Method for the Case of Many Subregions.

Bourgat, Glowinski, Le Tallec, Vidrascu (1989): Variational formulation and algorithm for trace operator in domain decomposition calculations (Neumann-Neumann Method)

Gander, Halpern, Nataf (1999): Optimal convergence for overlapping and non-overlapping Schwarz waveform relaxation

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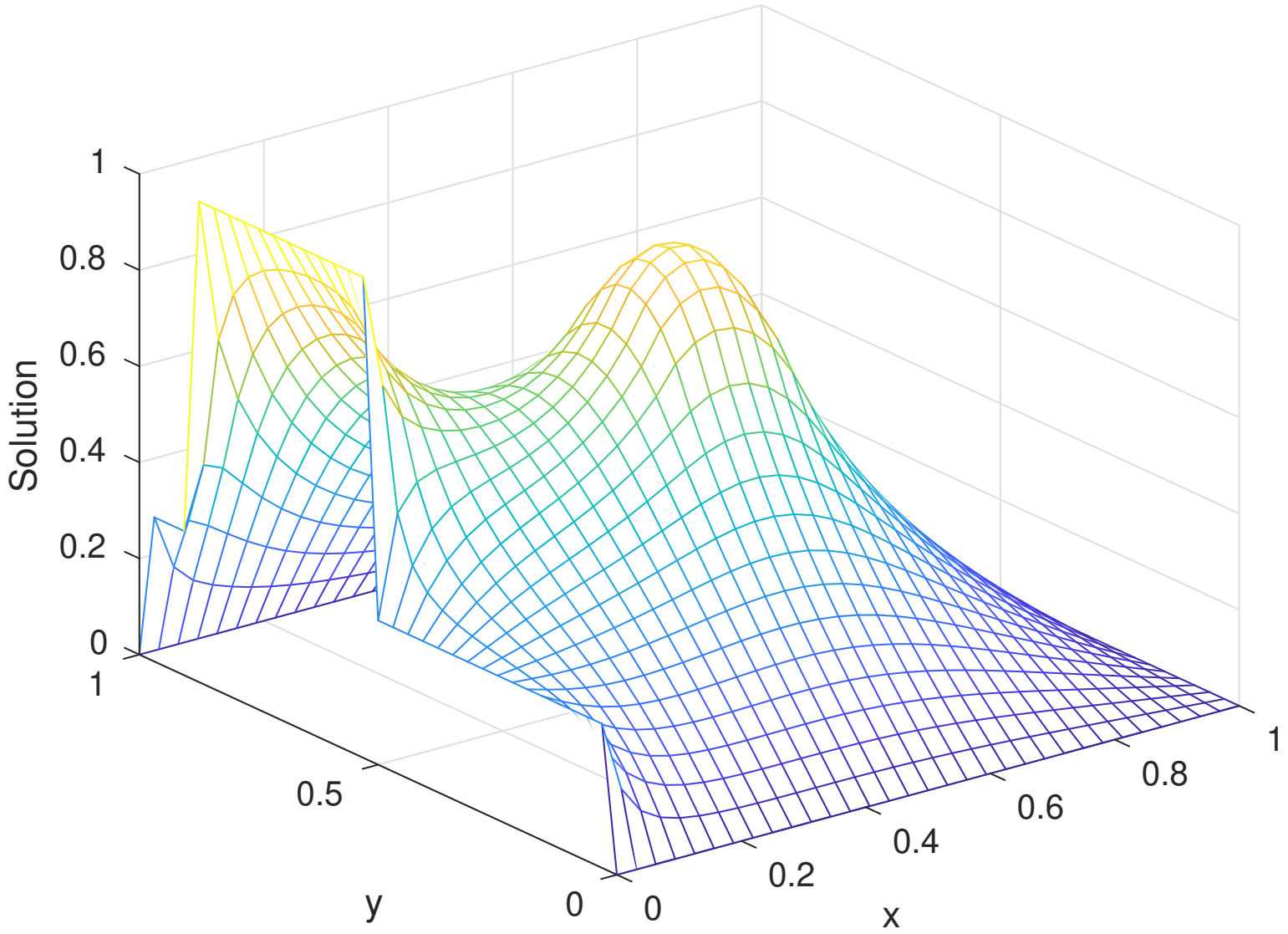
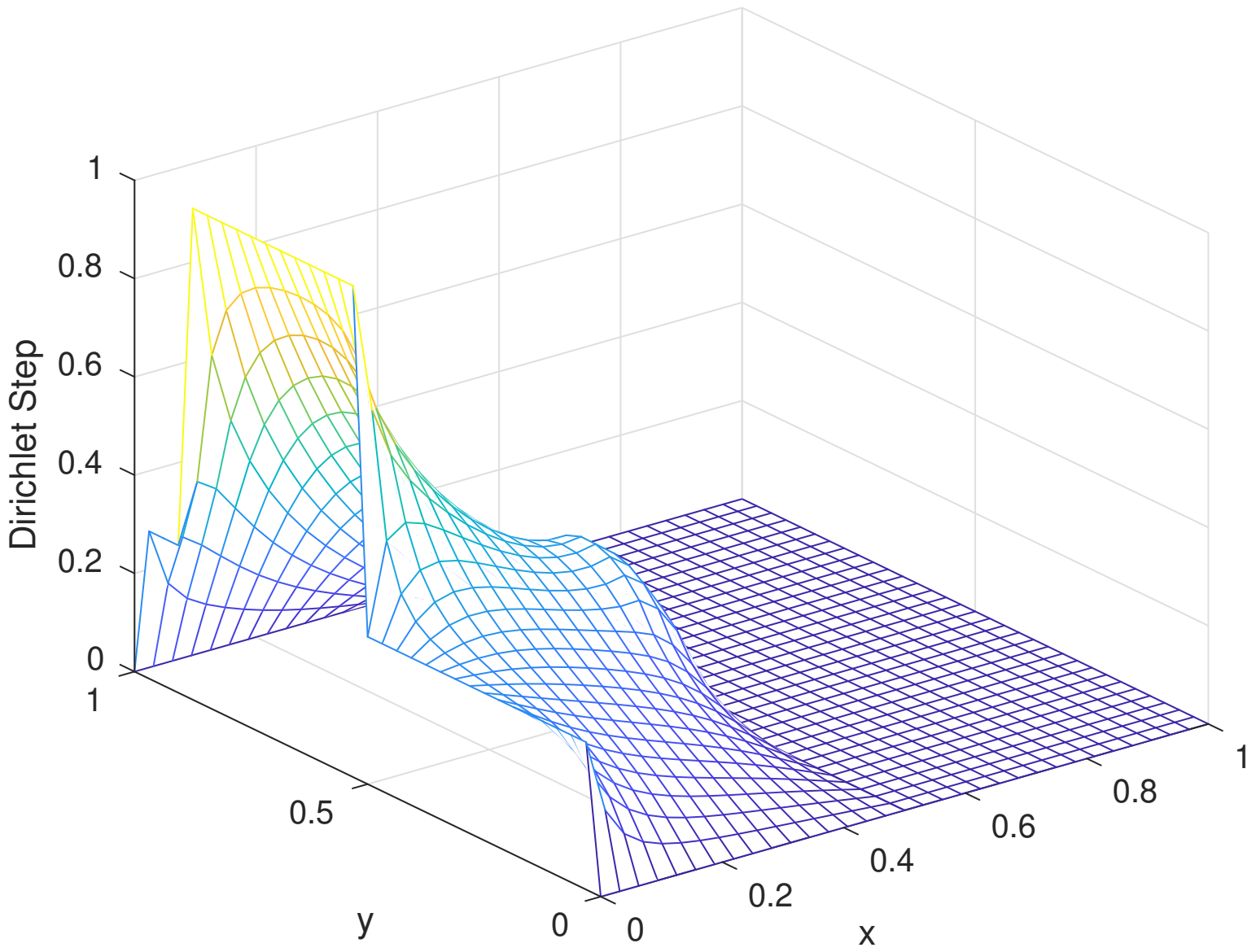
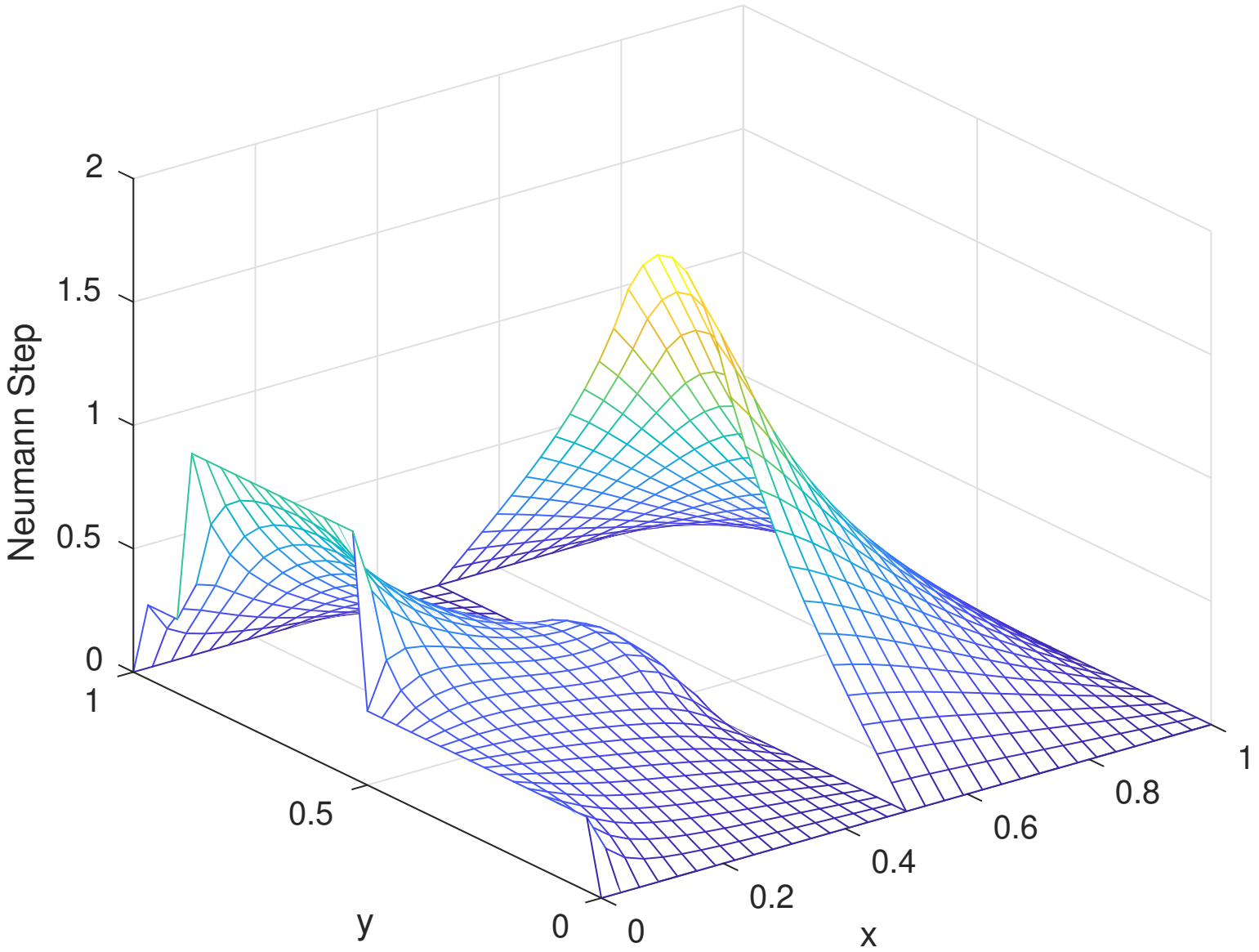


Dirichlet-Neumann iteration 1 left, symmetric Ω_i



Dirichlet-Neumann iteration 1 right, symmetric Ω_i



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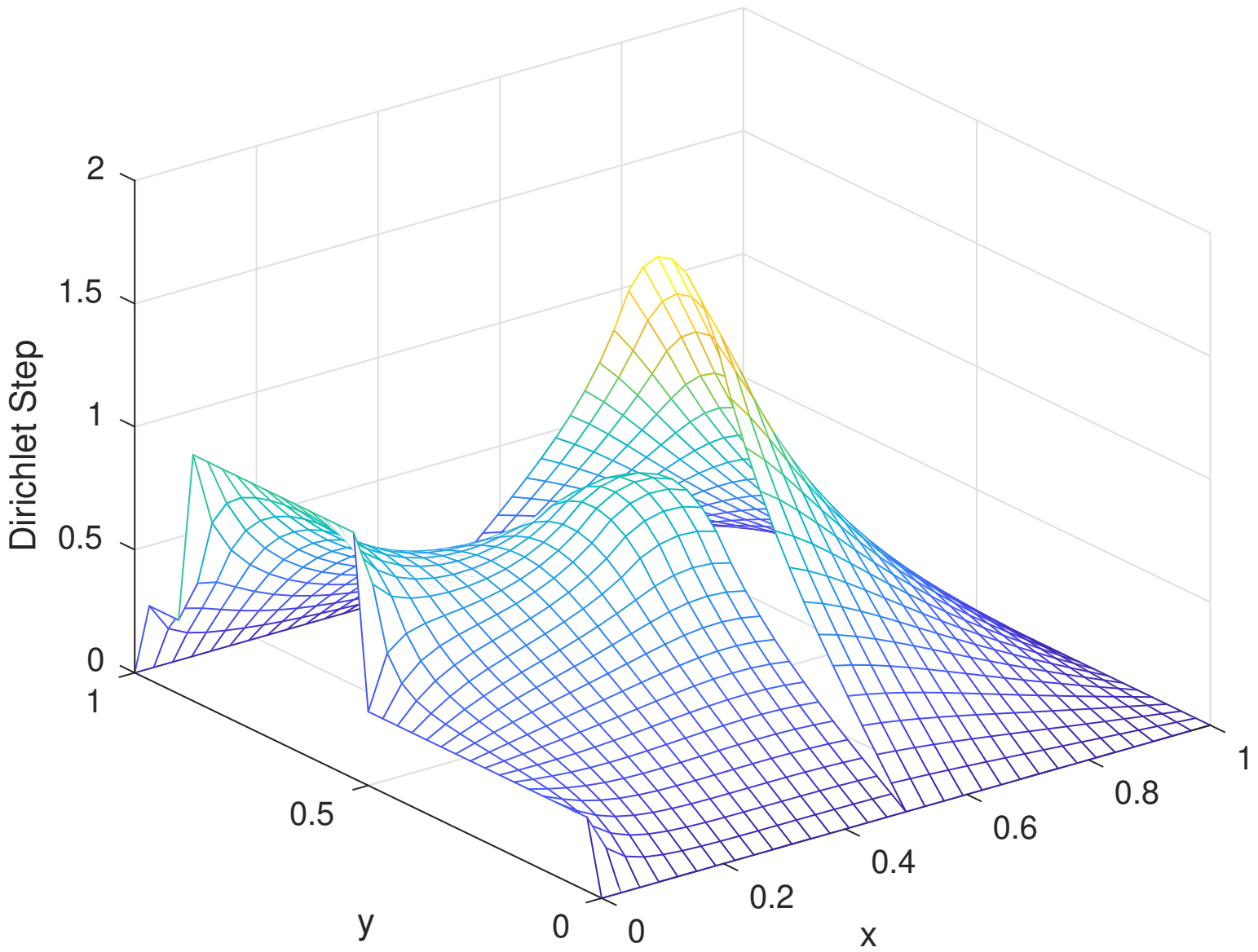
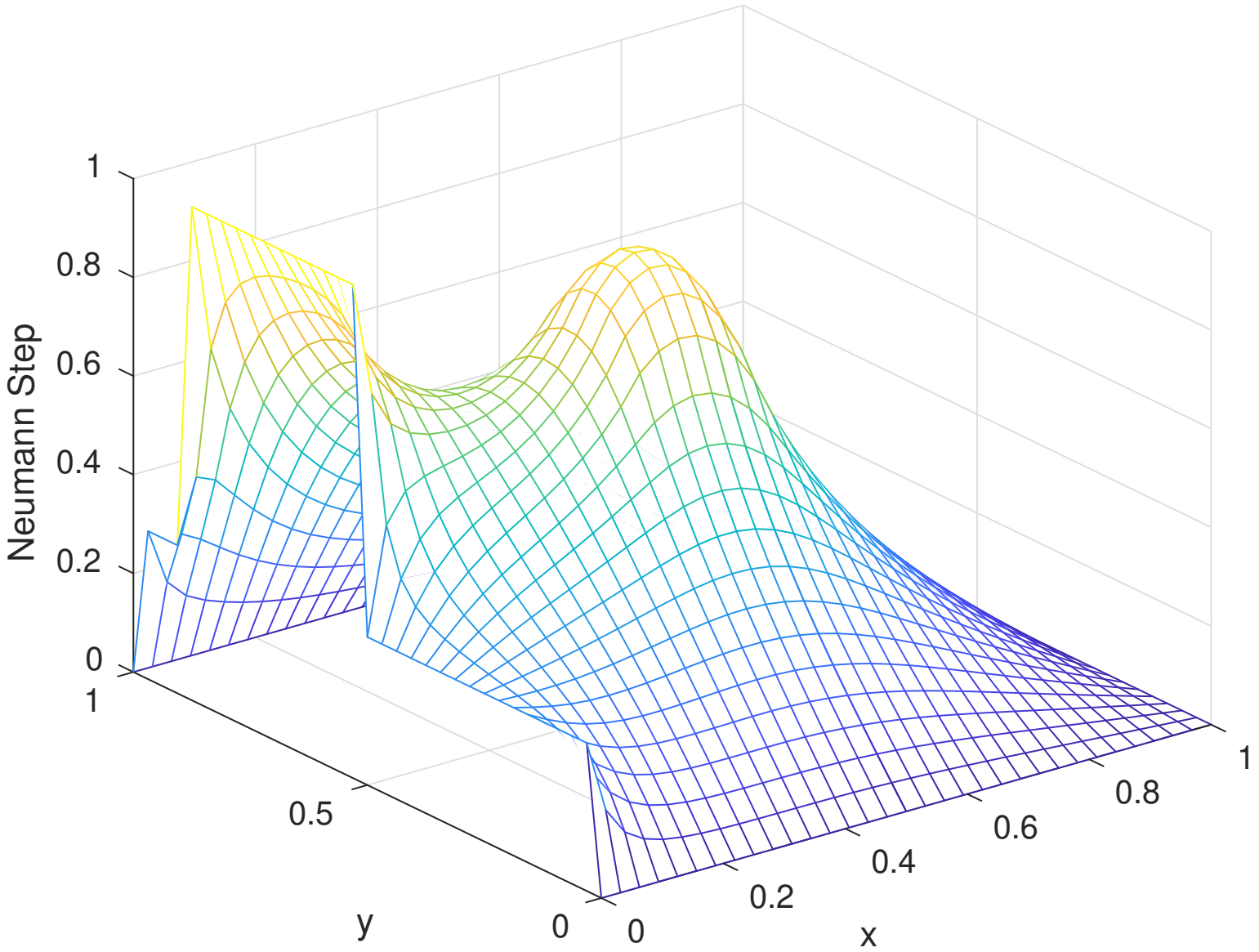
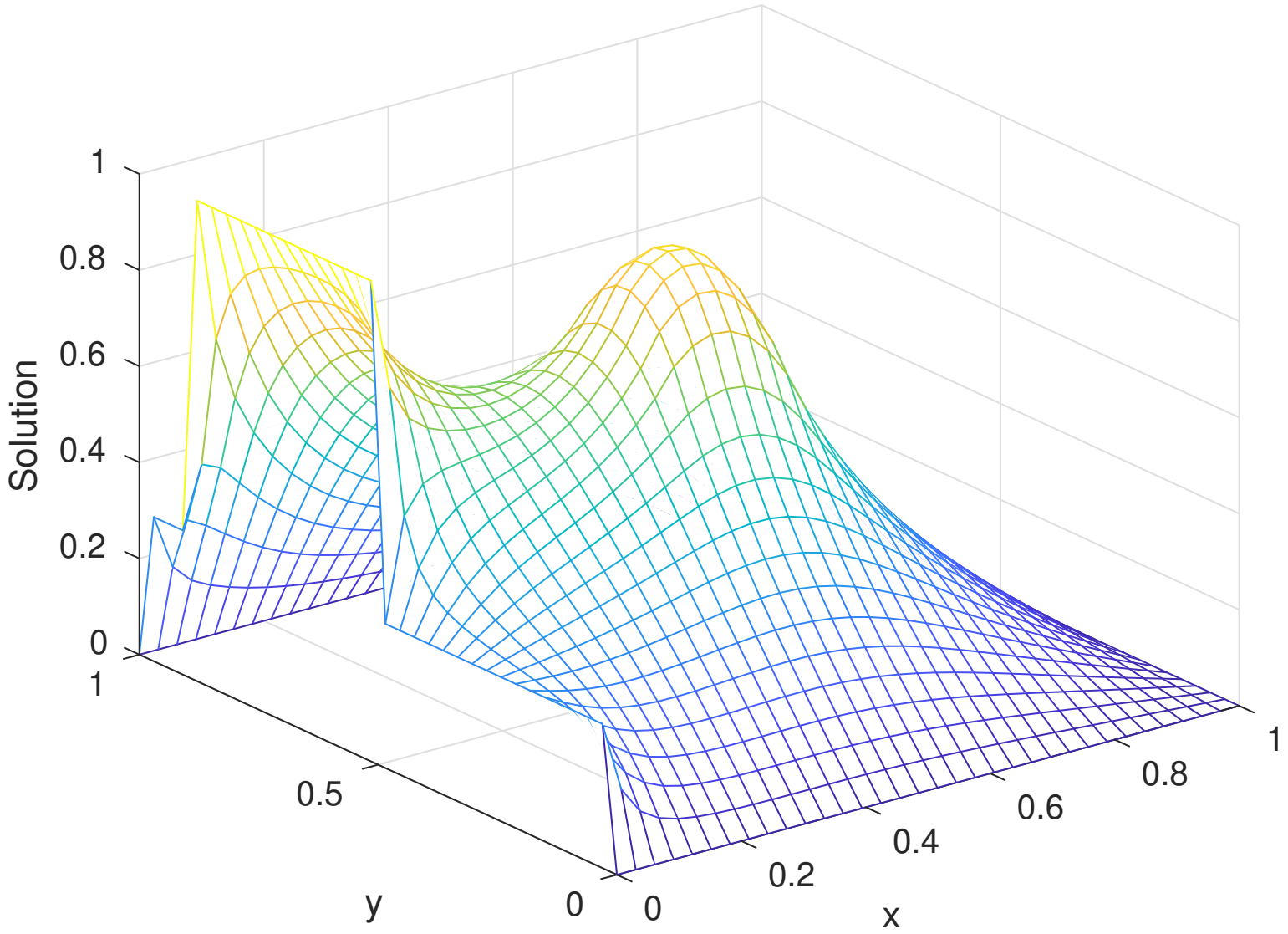
Dirichlet-Neumann iteration 2 left, symmetric Ω_i



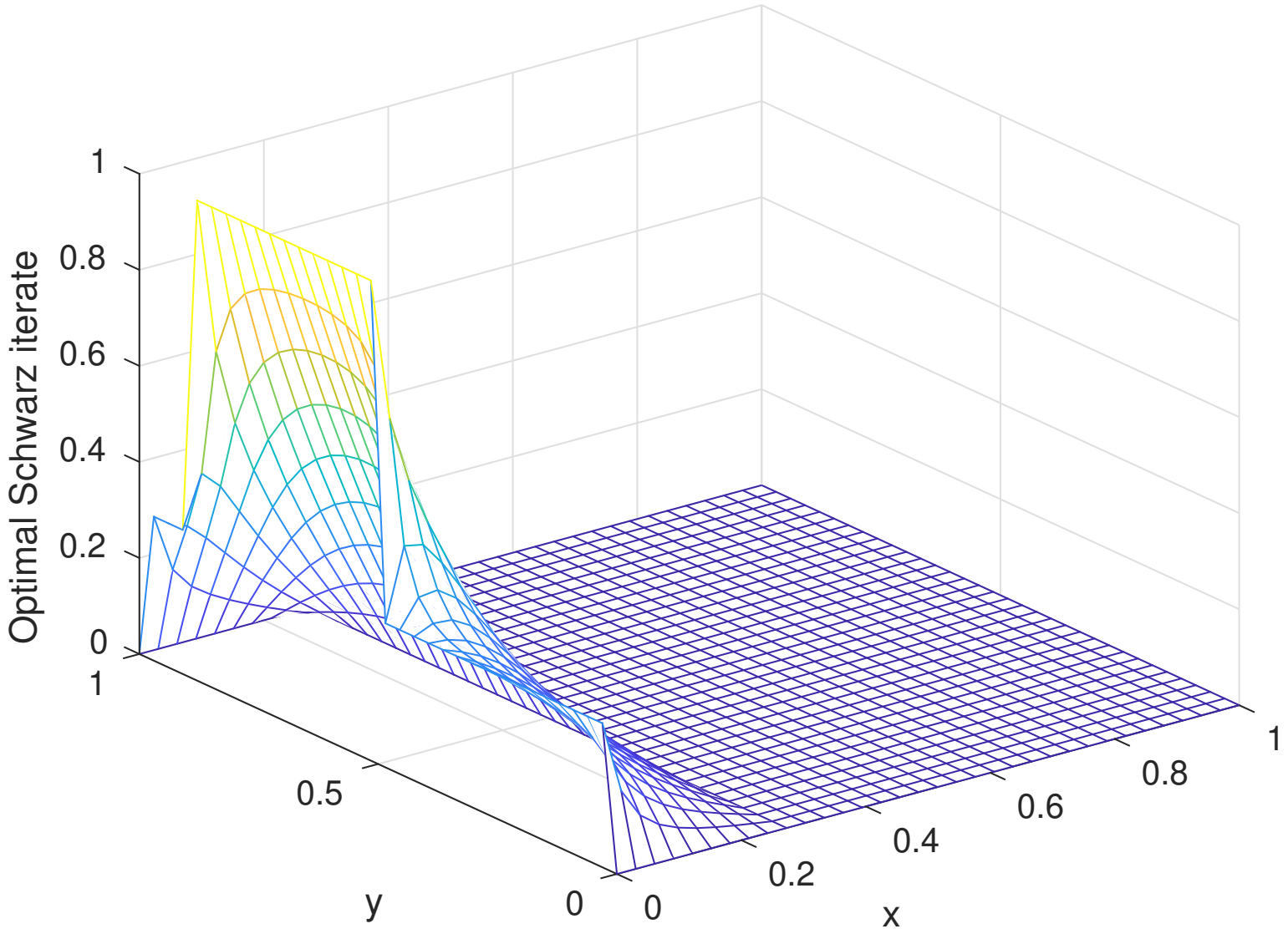
Dirichlet-Neumann iteration 2 right, symmetric Ω_i



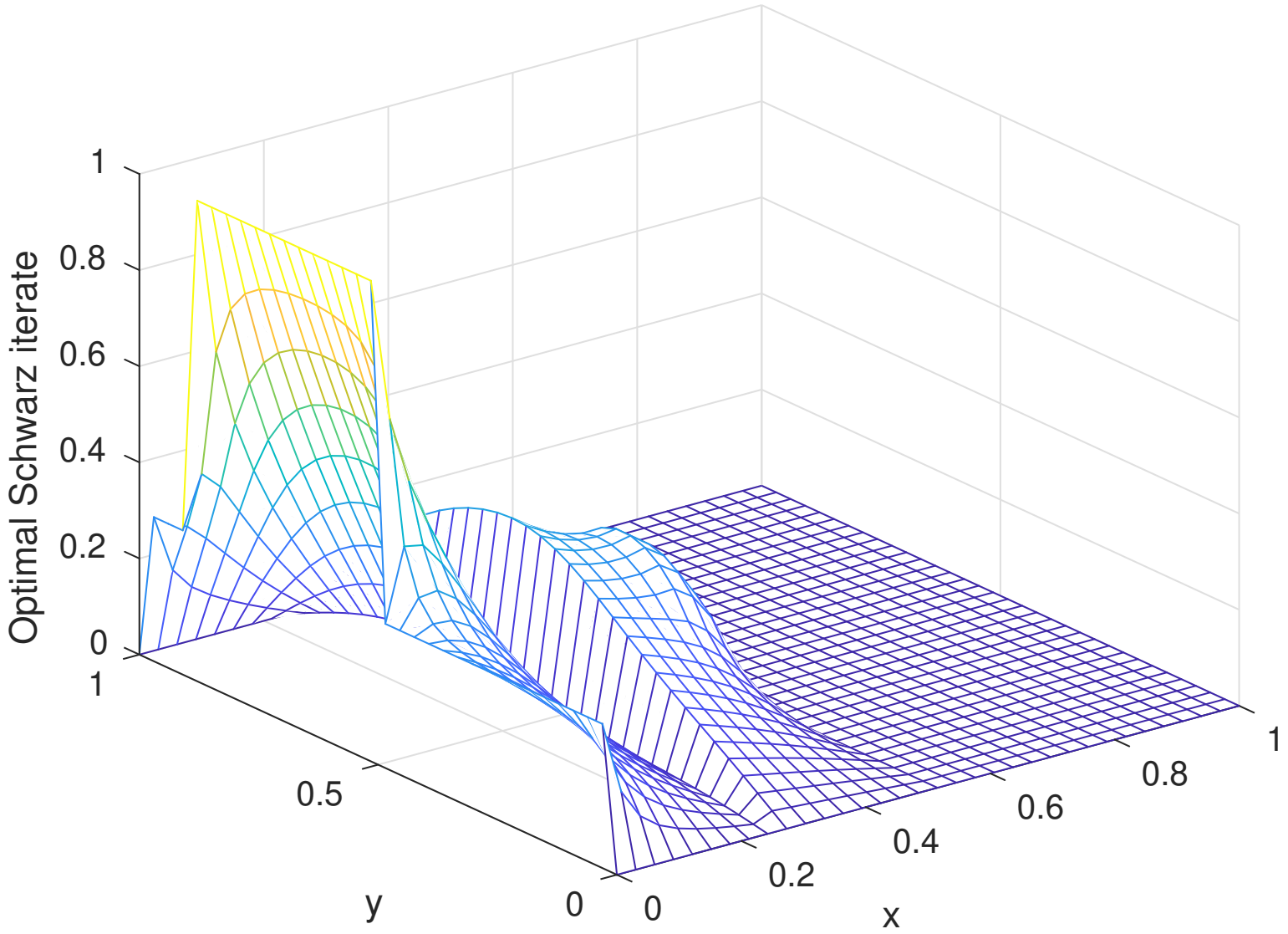
Optimal Schwarz Example:



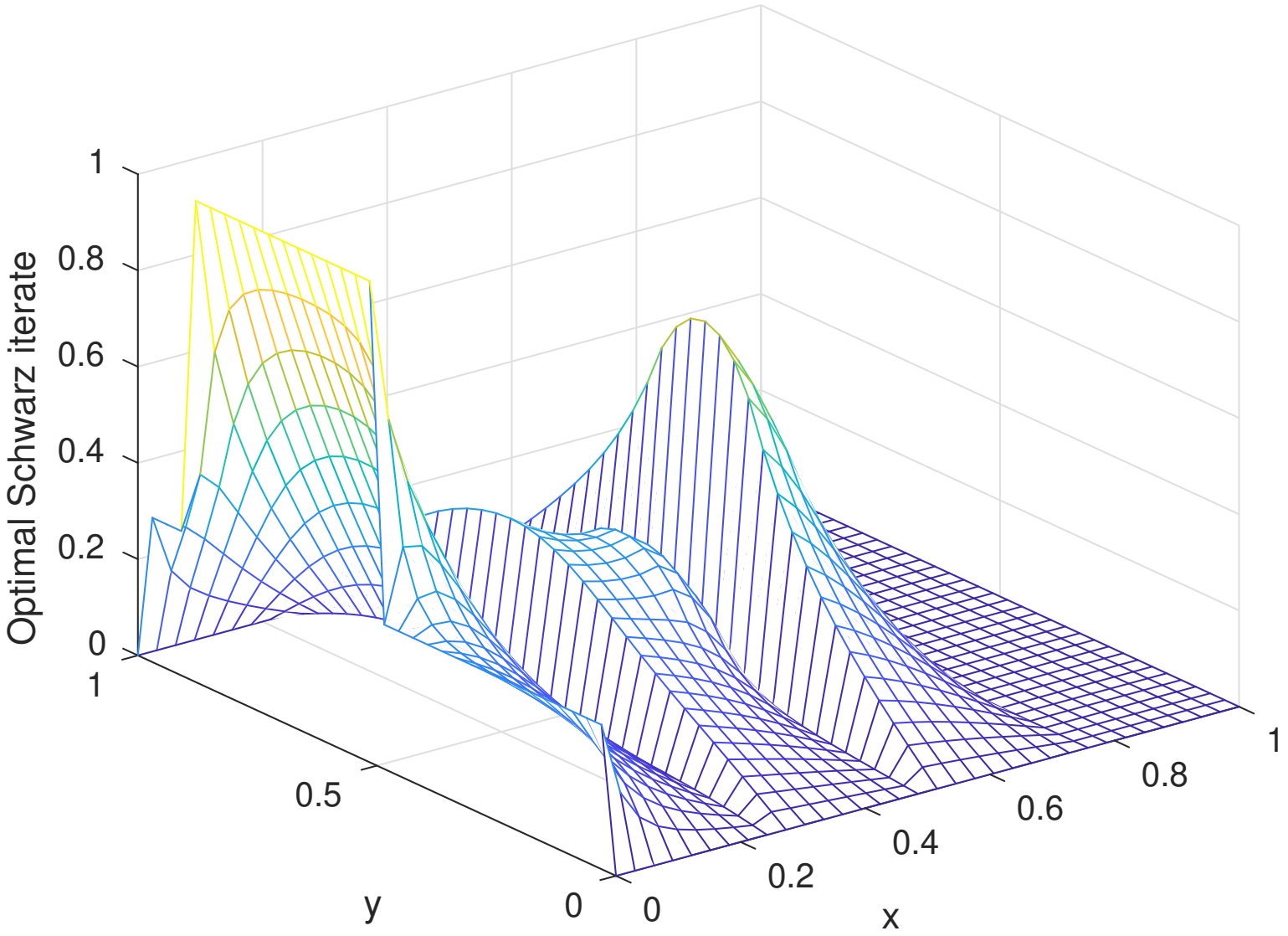
Optimal Sweeping Schwarz Iteration 1



Optimal Sweeping Schwarz Iteration 1



Optimal Sweeping Schwarz Iteration 1



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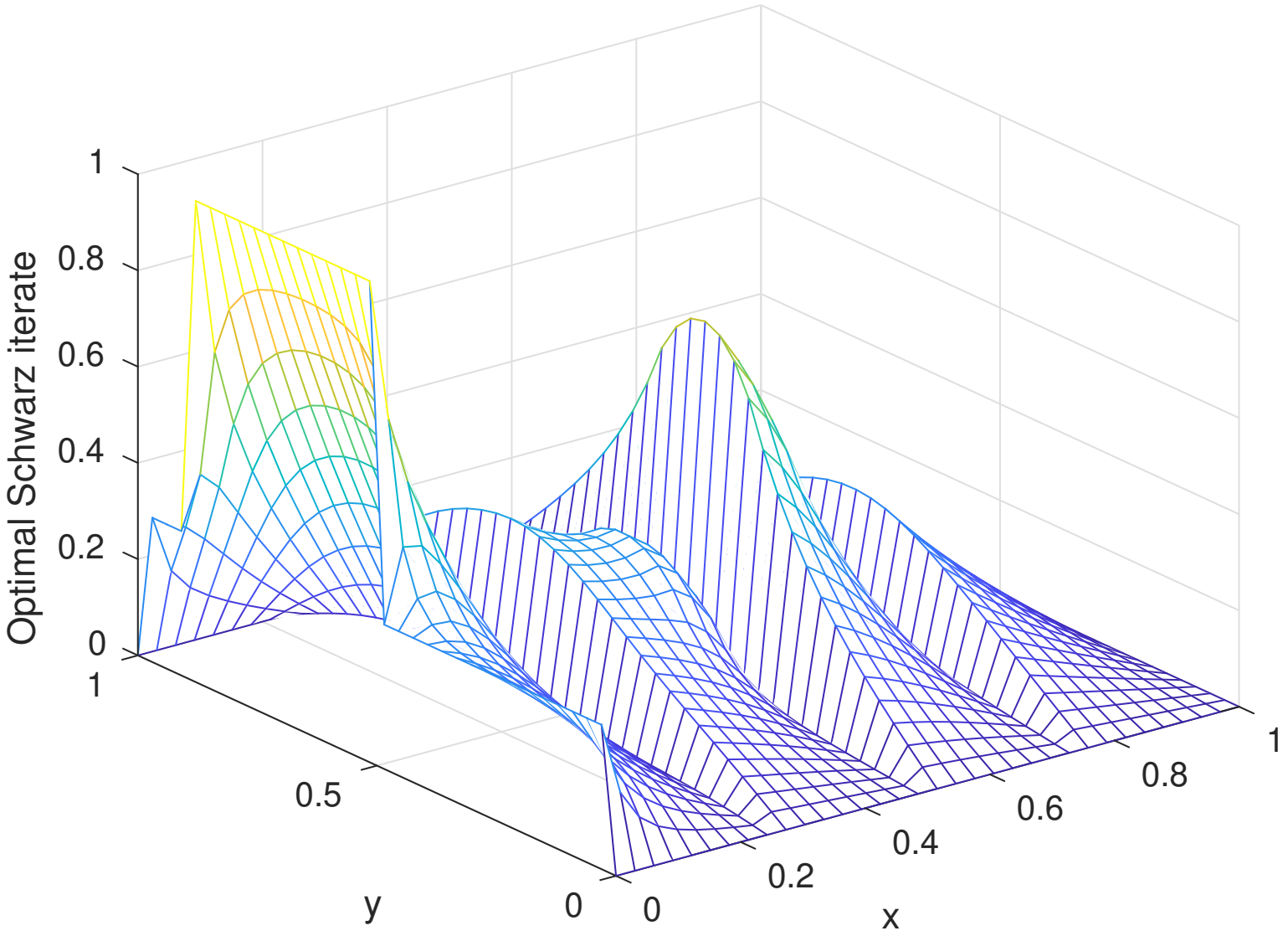

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Jacobi, Seidel
Example
Modern Notation
SOR

Extrapolation
Richardson
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Krylov Methods
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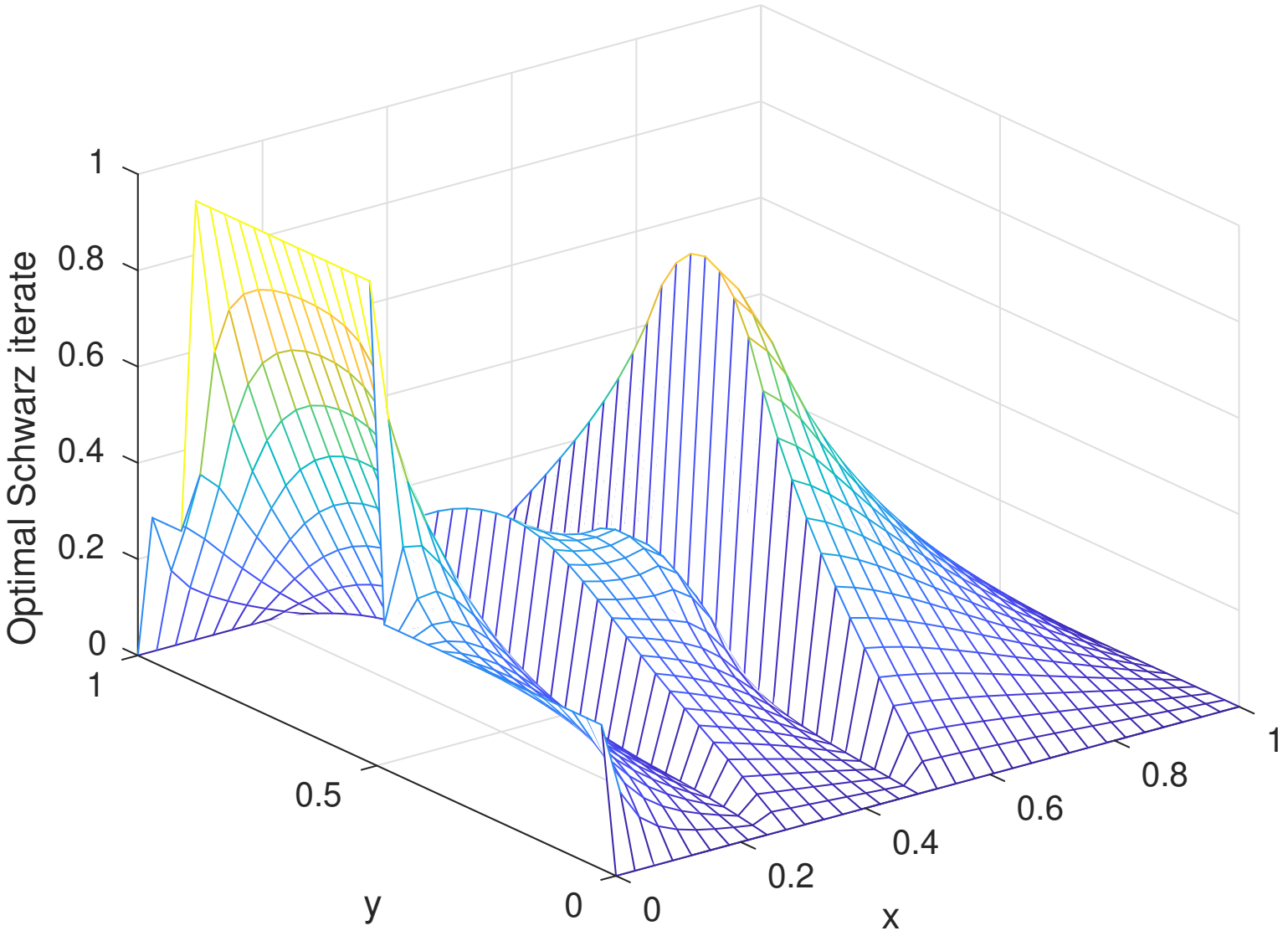
Example
Paige, Saunders, Saad, Freund, Van Der Vorst

Domain Decomposition
Invention of Schwarz
Schwarz example
Further DD Methods
Example
Optimal Schwarz

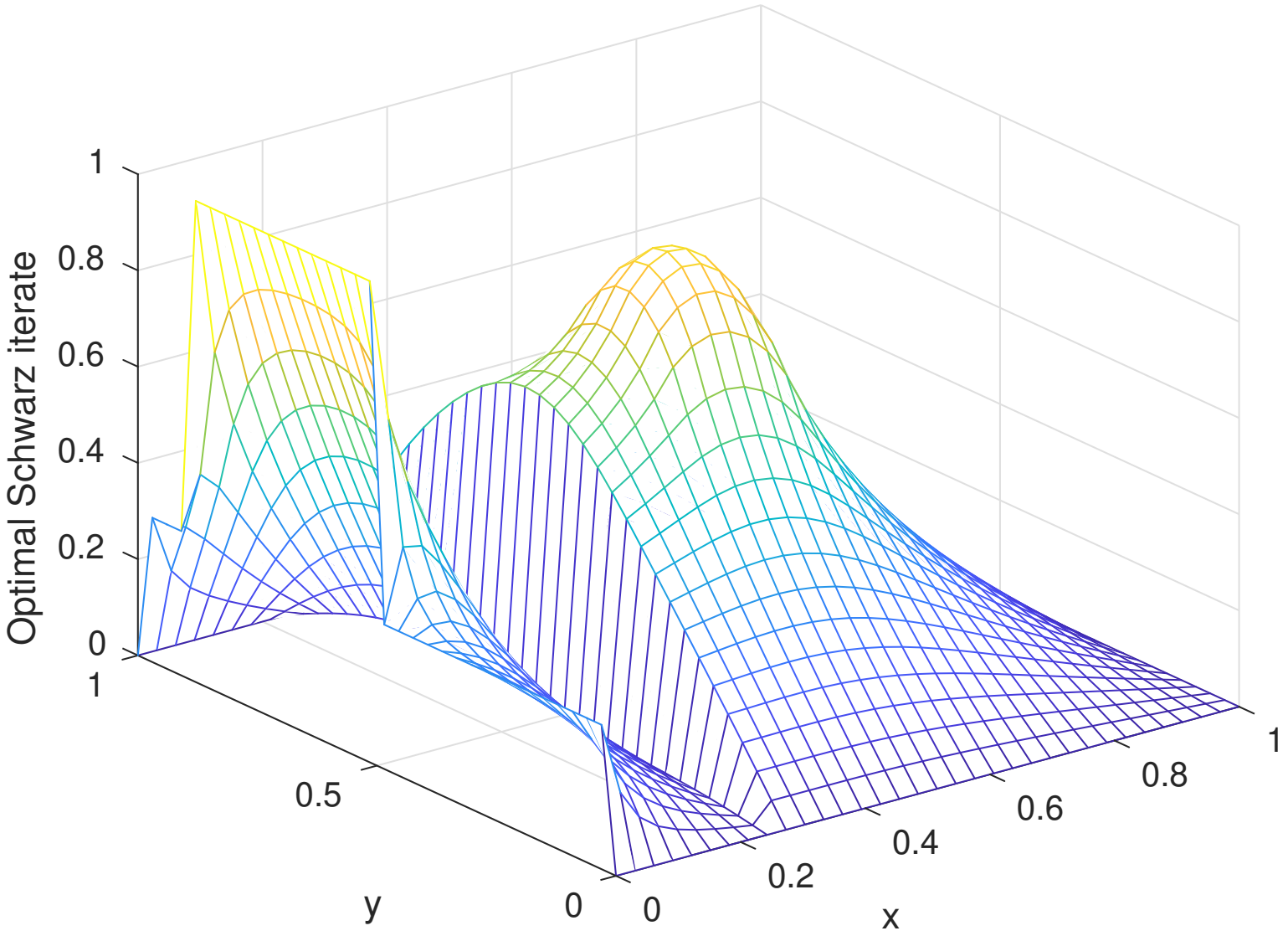
Multigrid
MG example
Fedorenko

Preconditioning

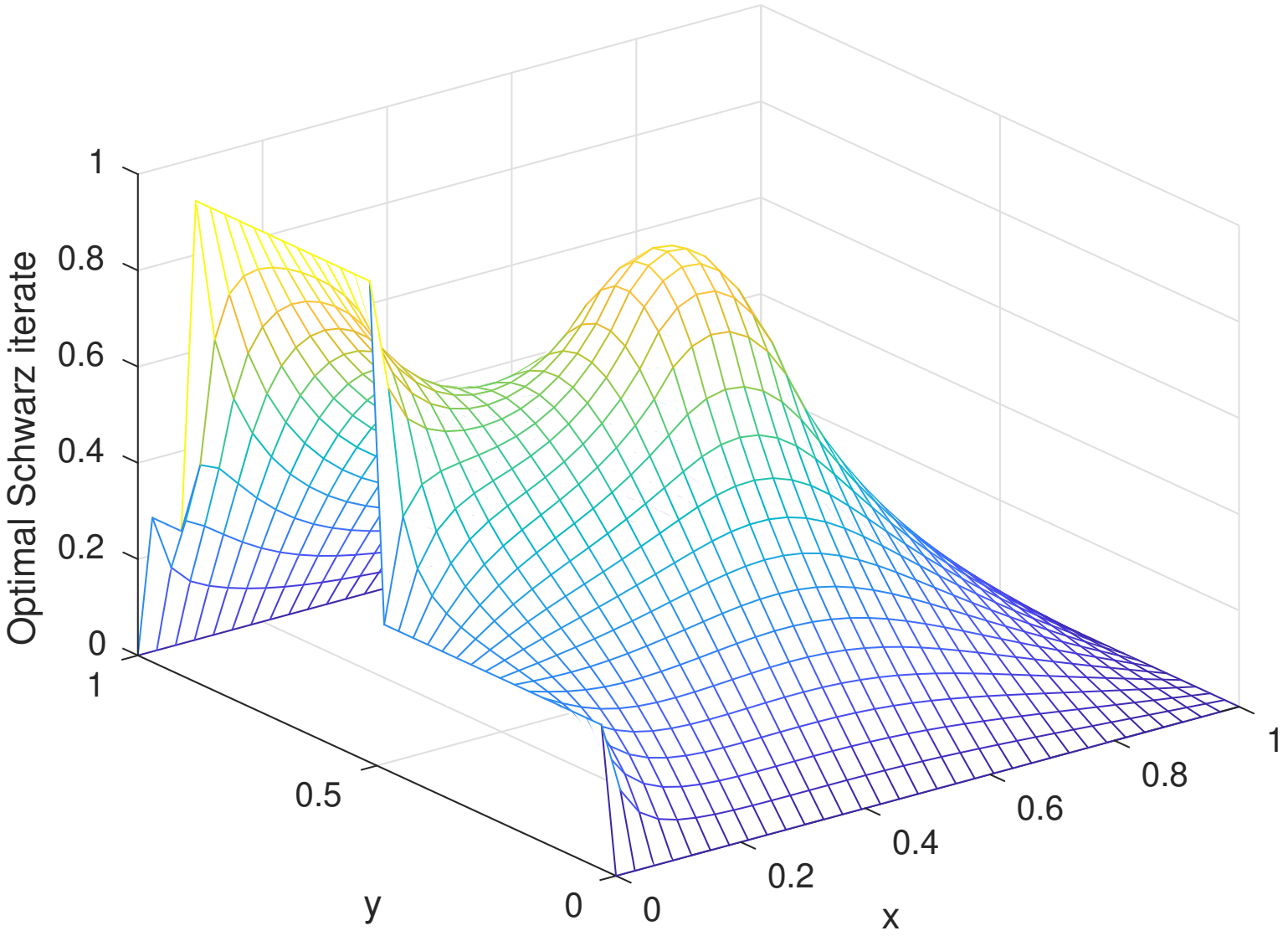
Optimal Sweeping Schwarz Iteration 1



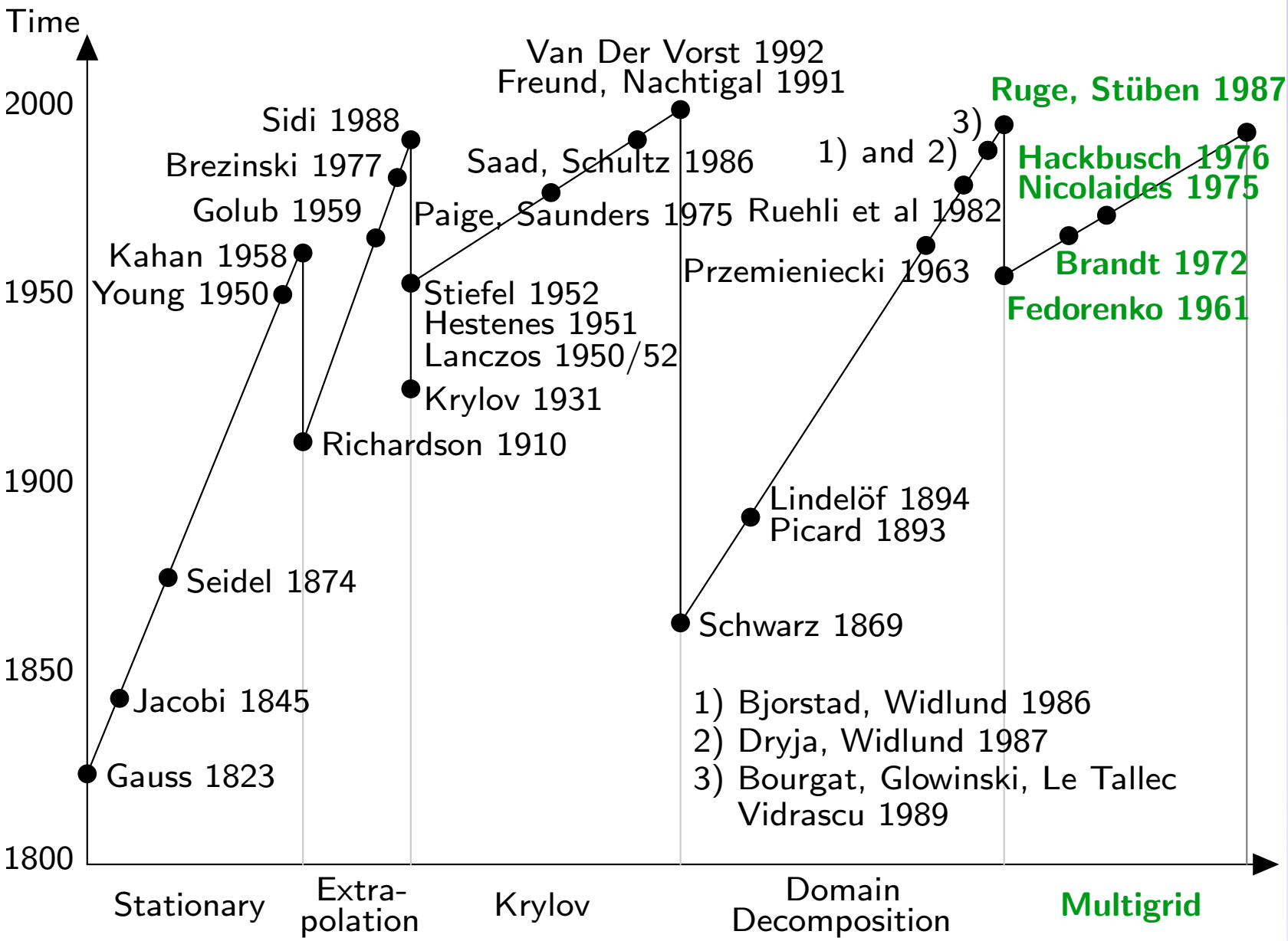
Optimal Sweeping Schwarz Iteration 1



Optimal Sweeping Schwarz Iteration 1



Iterative Methods for Linear Systems



Iterative Methods

Martin J. Gander

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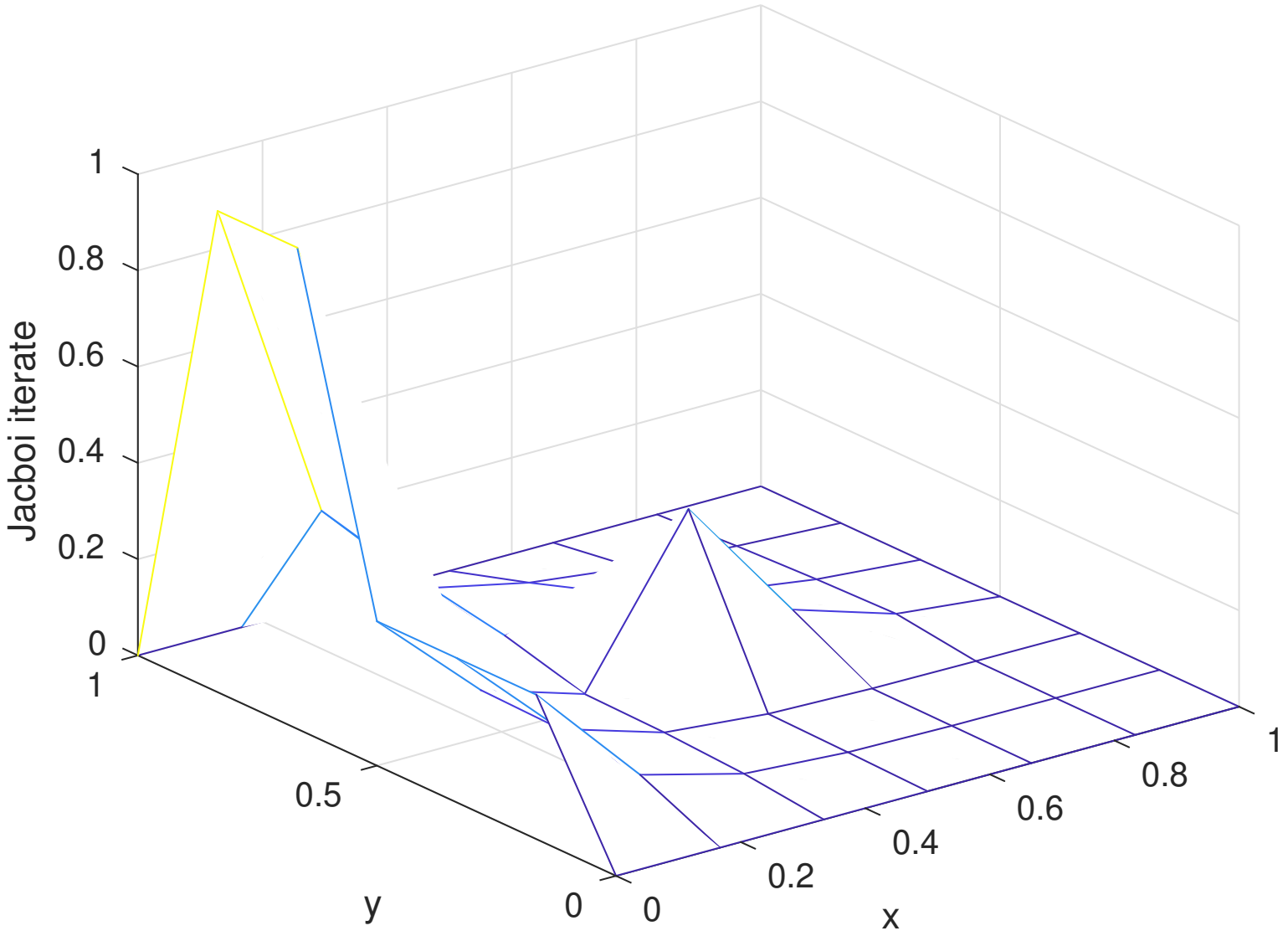
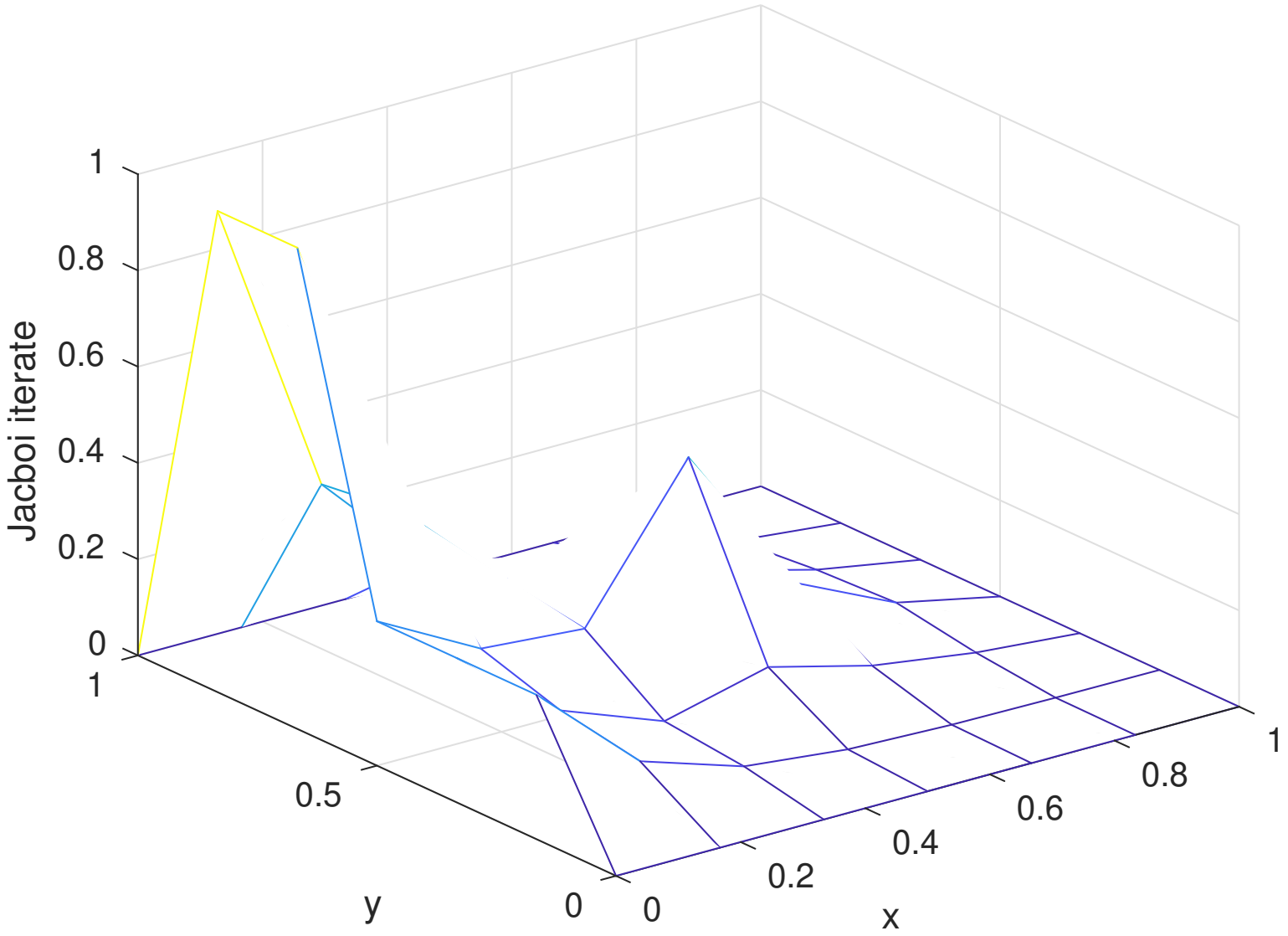
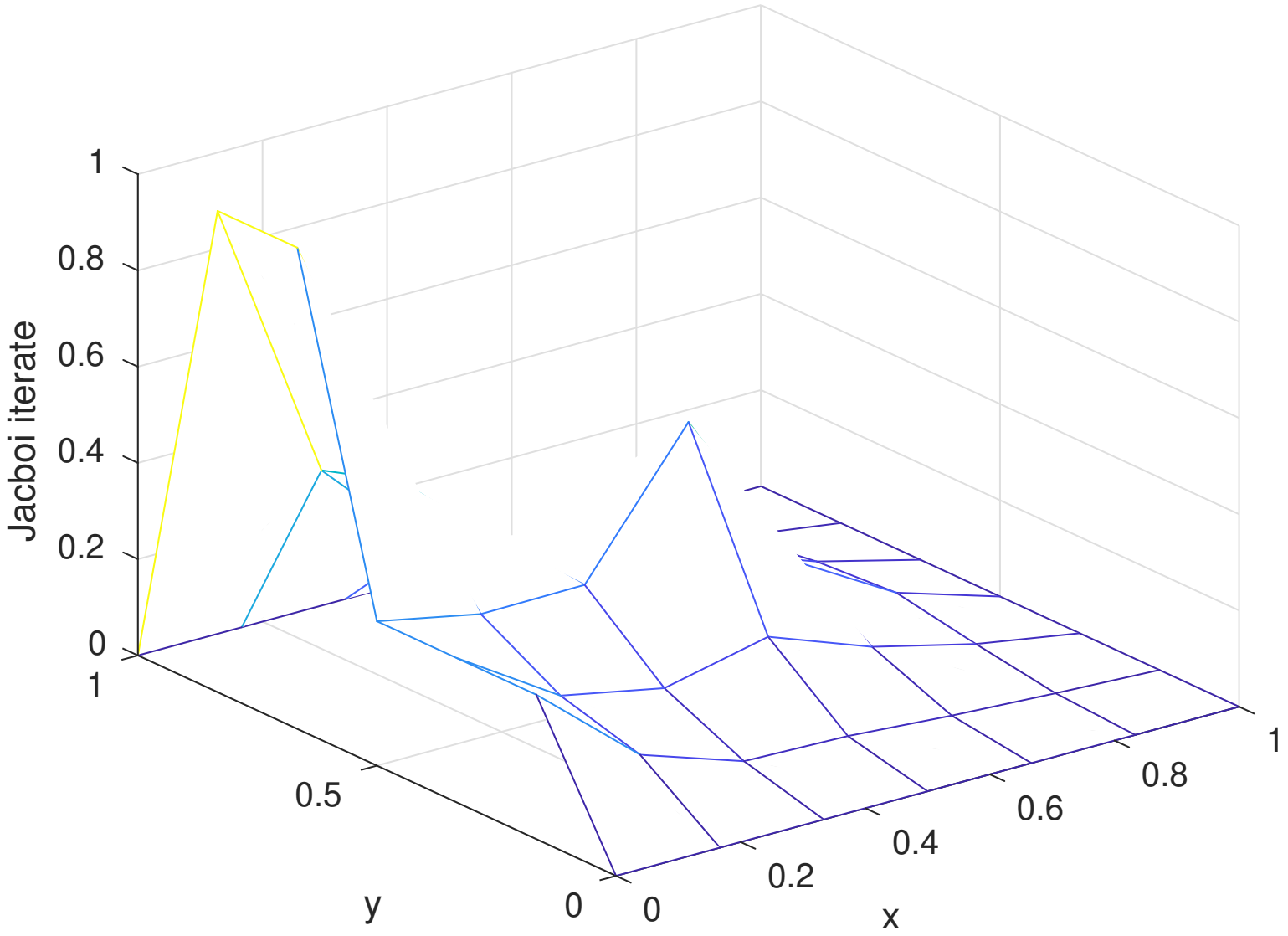
Coarse Gauss-Seidel iteration 1



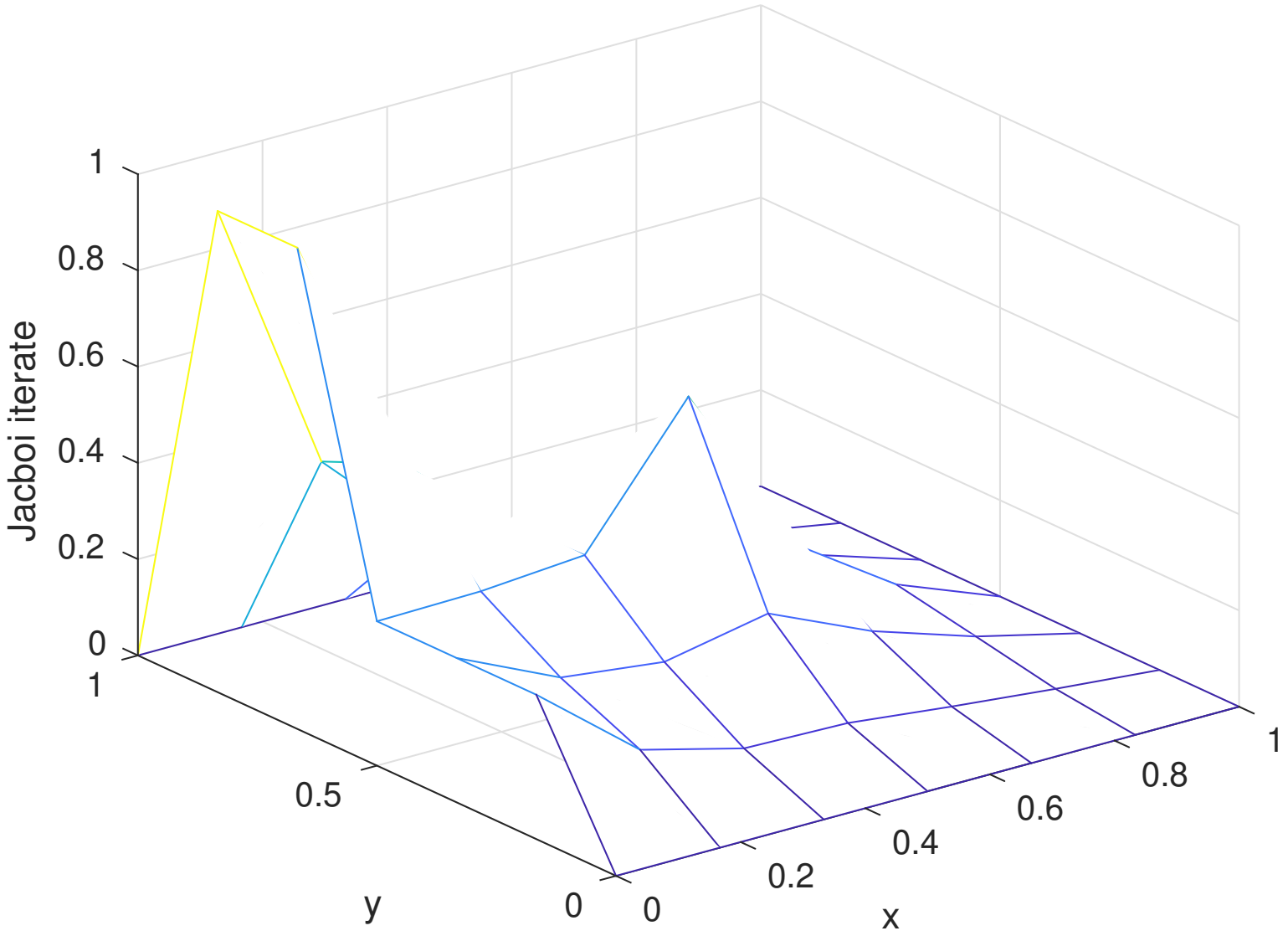
Gauss-Seidel iteration 2



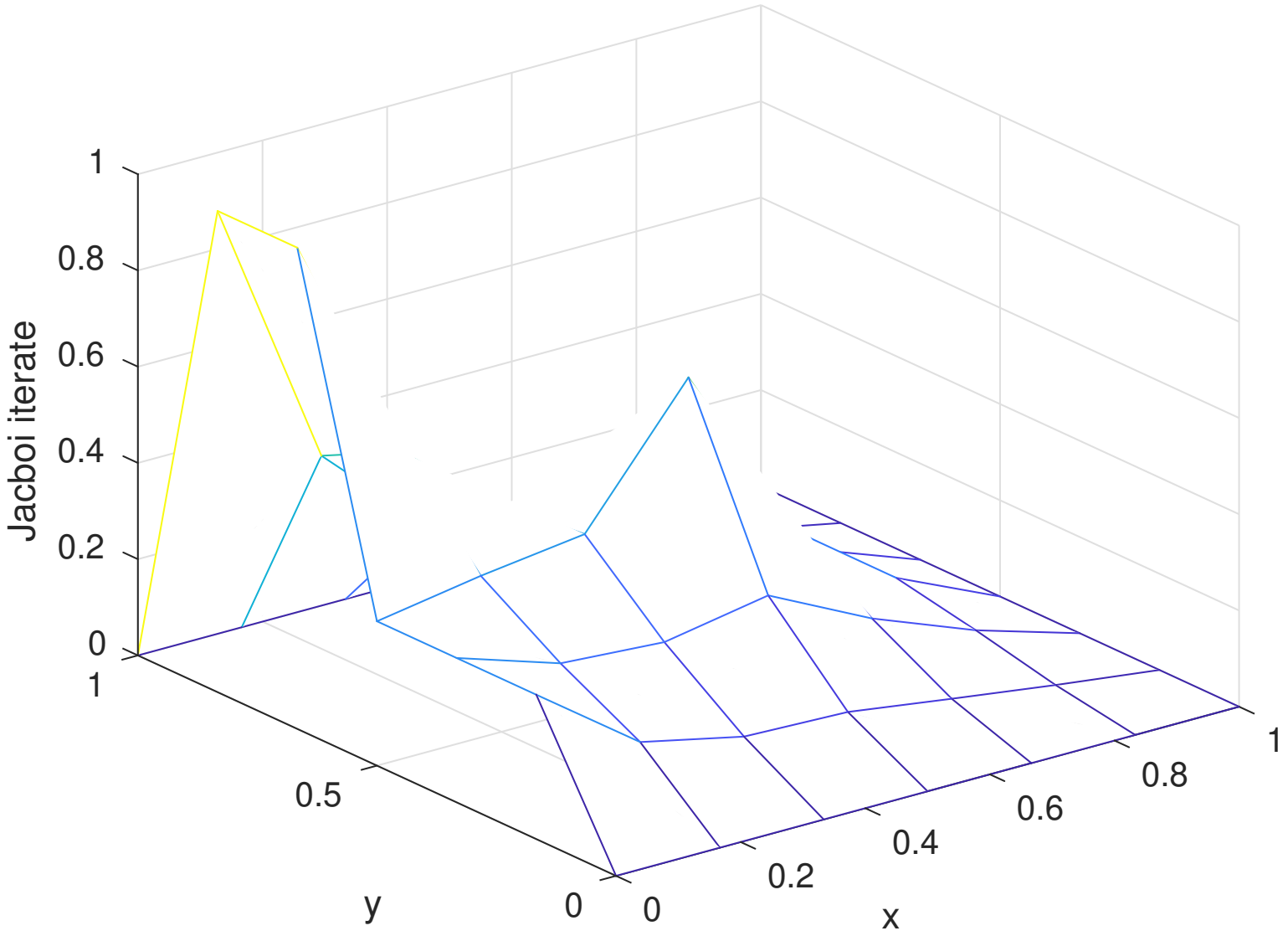
Gauss-Seidel iteration 3



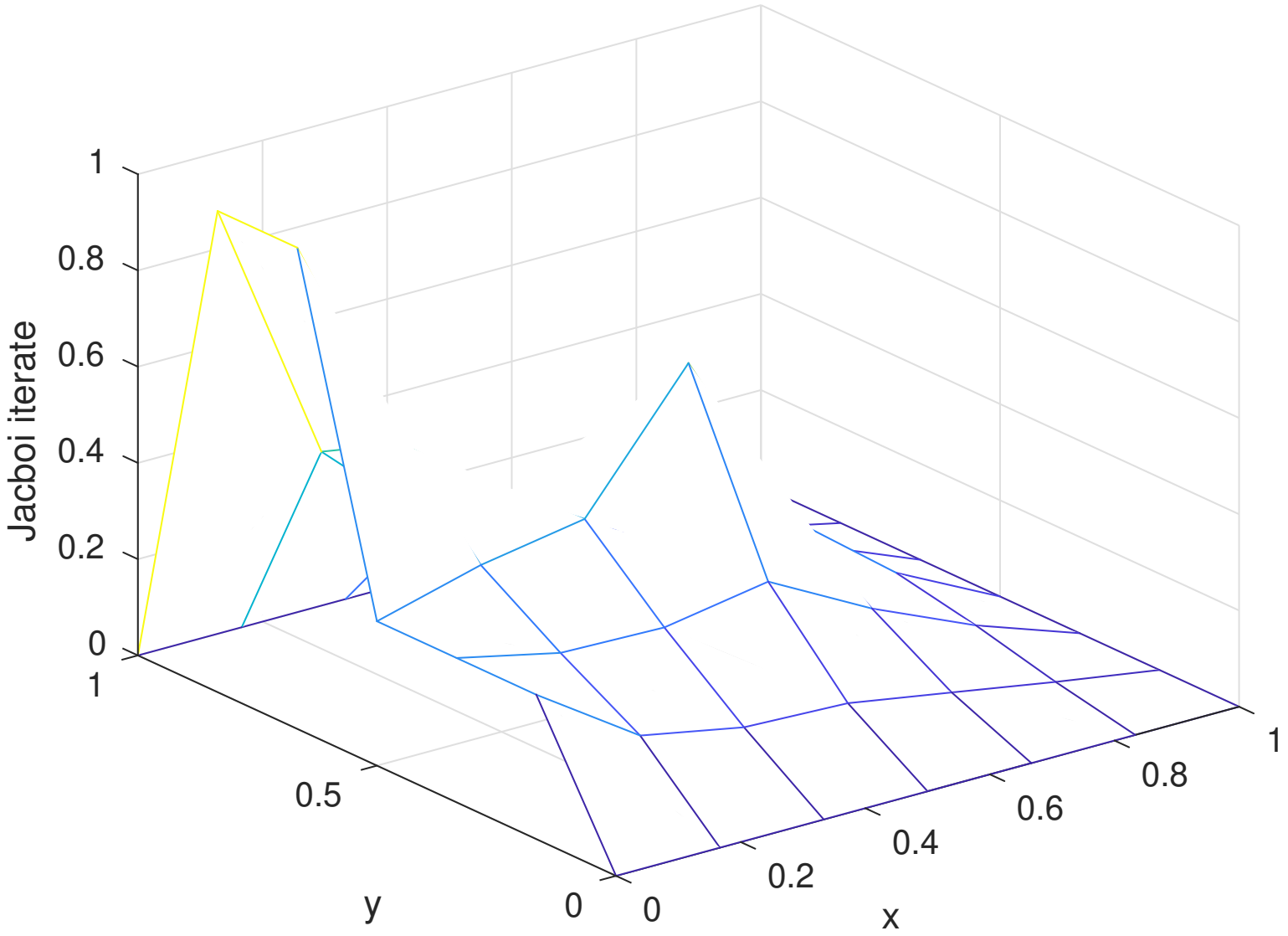
Gauss-Seidel iteration 4



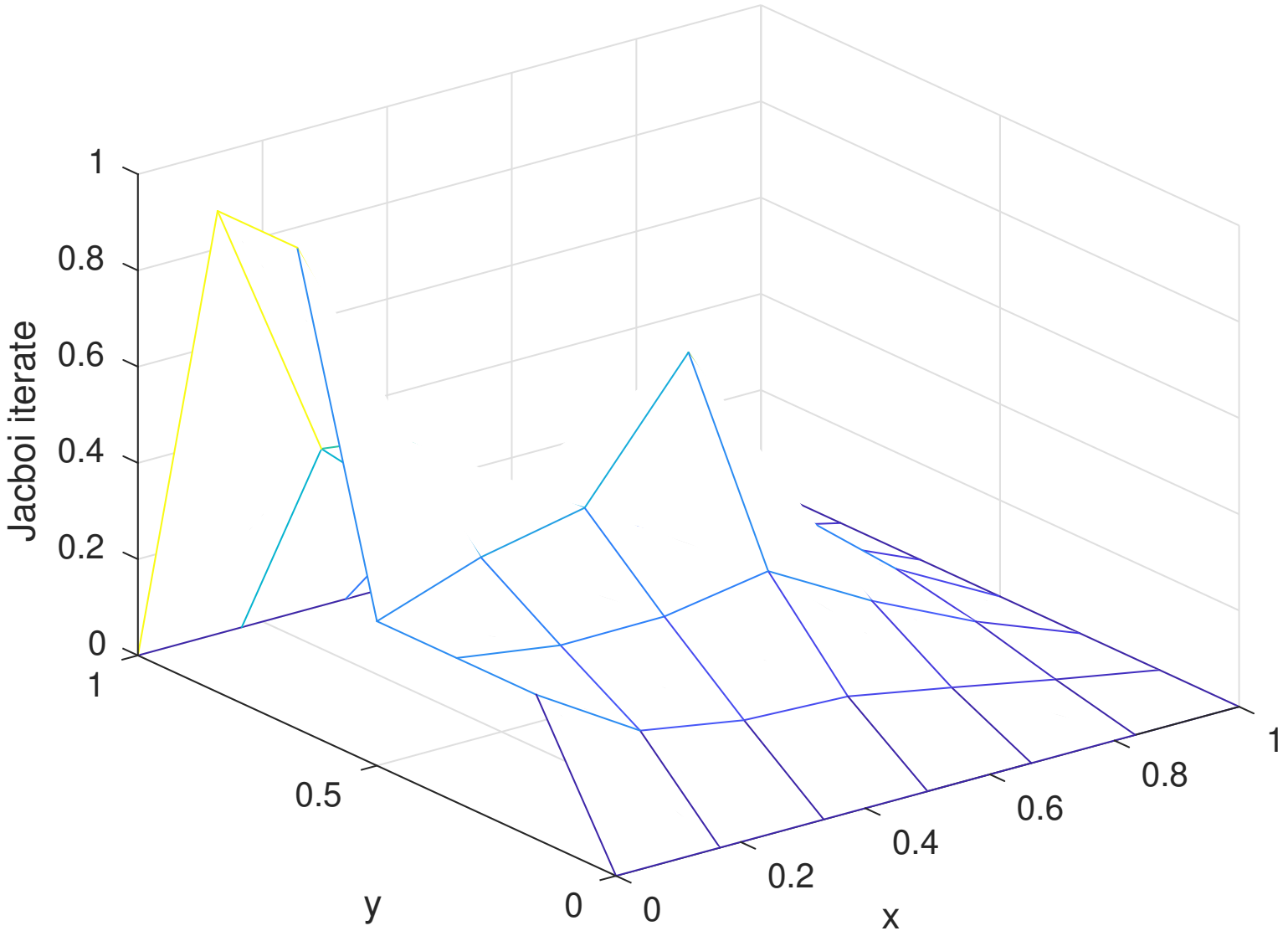
Gauss-Seidel iteration 5



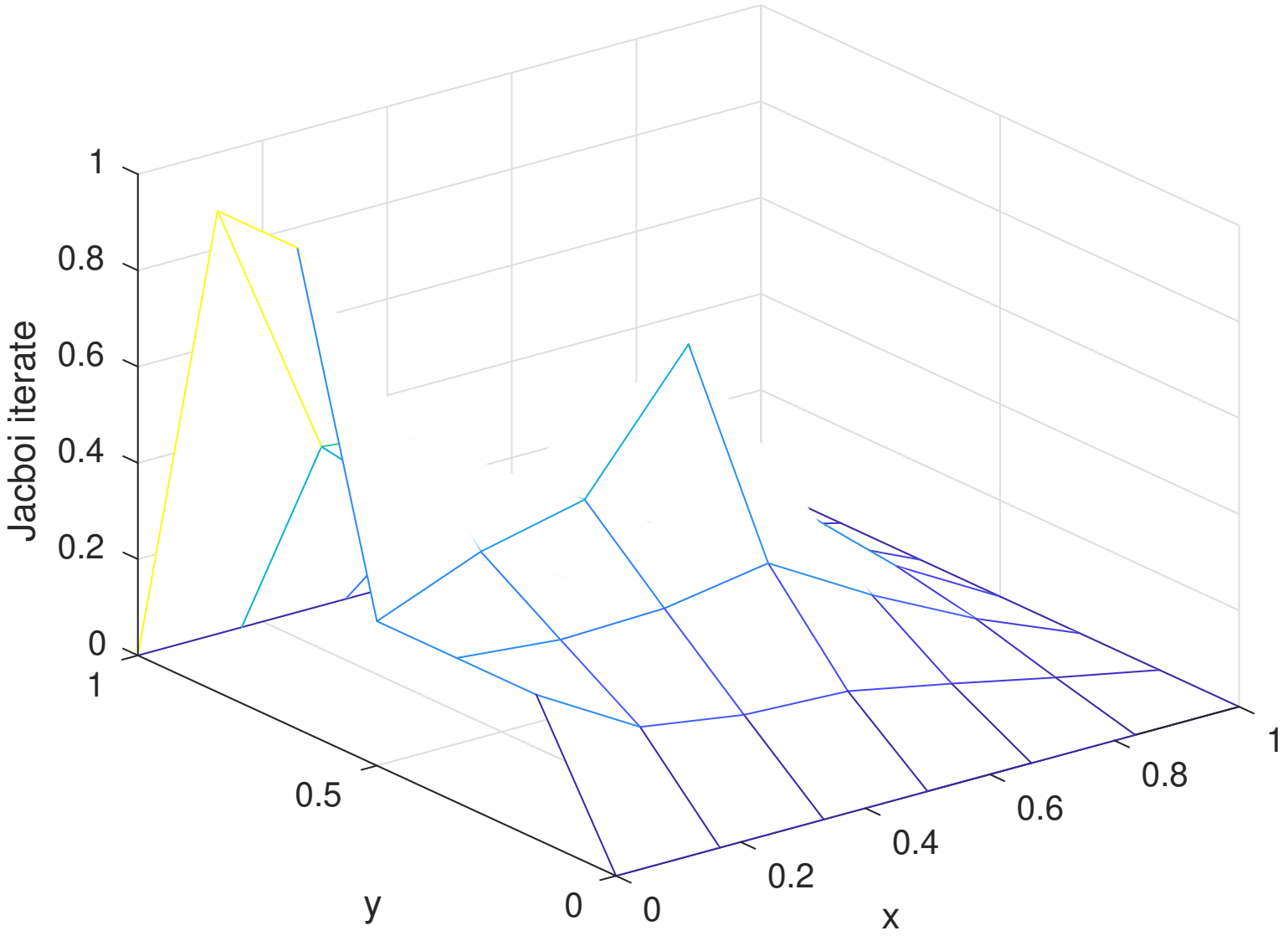
Gauss-Seidel iteration 6



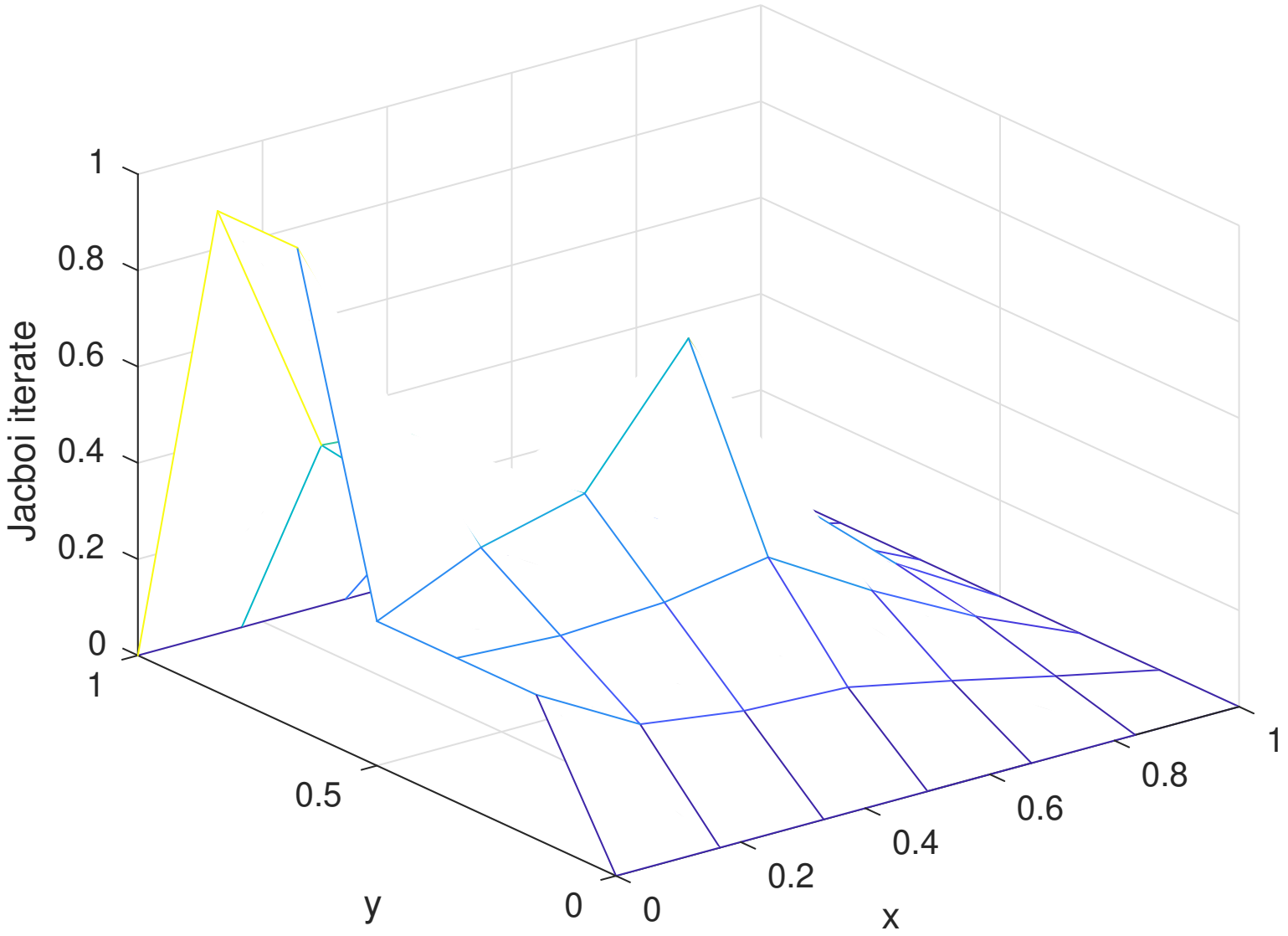
Gauss-Seidel iteration 7



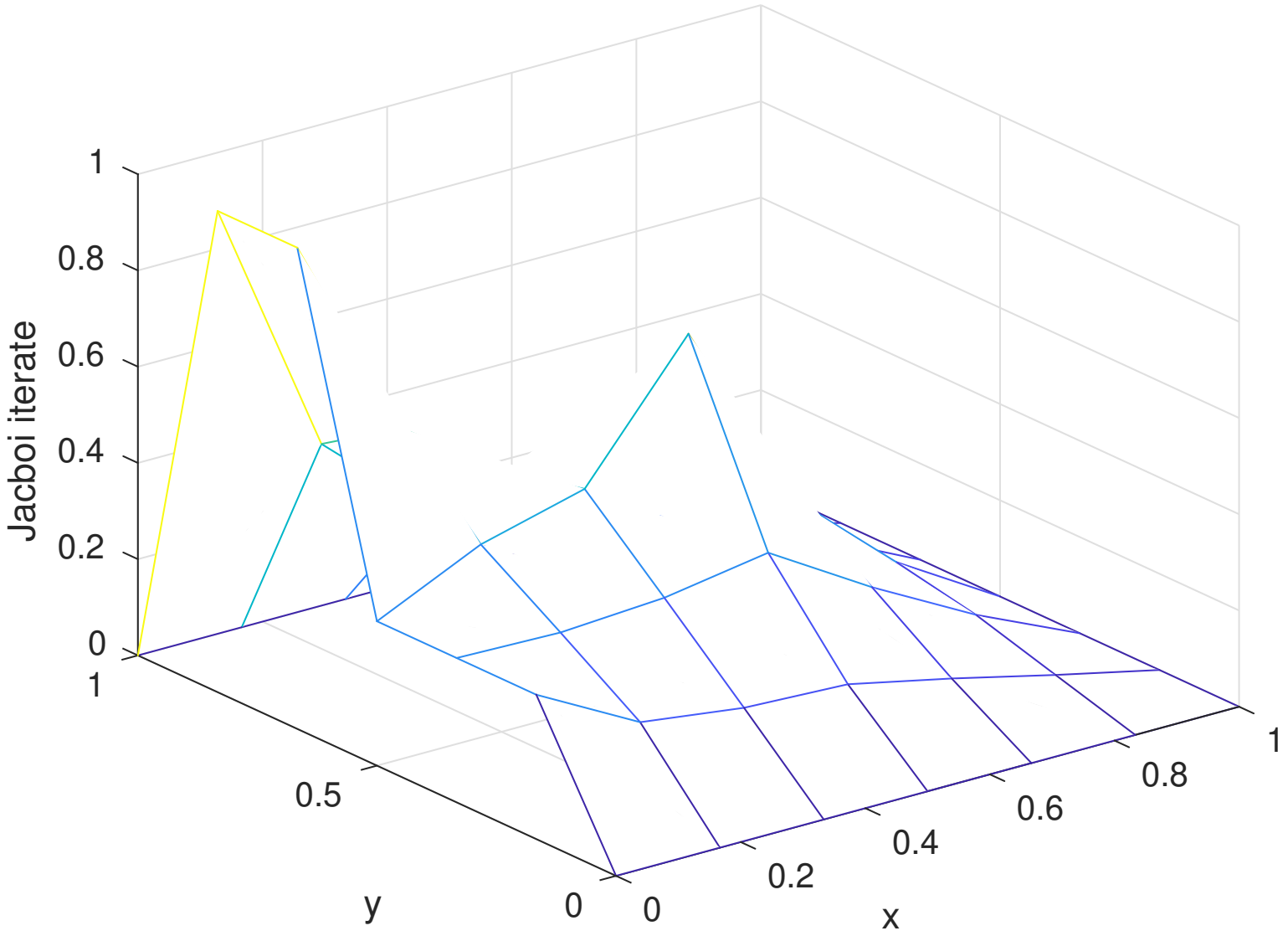
Gauss-Seidel iteration 8



Gauss-Seidel iteration 9



Gauss-Seidel iteration 10



The Idea of Multigrid

To solve a linear system

$$A\boldsymbol{u} = \boldsymbol{f}$$

which represents a discretized PDE, starting with an initial guess $\boldsymbol{u}^0$ one computes

$$\begin{aligned}
\boldsymbol{u}^n &= S(A, \boldsymbol{u}^n, \boldsymbol{f}, \nu_1)\,; \\
\boldsymbol{u}^n &= \boldsymbol{u}^n + PA_c^{-1}R(\boldsymbol{f} - A\boldsymbol{u}^n)\,; \\
\boldsymbol{u}^{n+1} &= S(A, \boldsymbol{u}^n, \boldsymbol{f}, \nu_2)\,;
\end{aligned}$$

- Smoother $S(A, \boldsymbol{u}^n, \boldsymbol{f}, \nu_1)$ (Jacobi, Gauss-Seidel)
- Prolongation P (interpolation)
- Restriction $R = P^T$
- Coarse matrix $A_c = RAP$ (Galerkin)

$\Longrightarrow$ **This idea is then applied recursively !**

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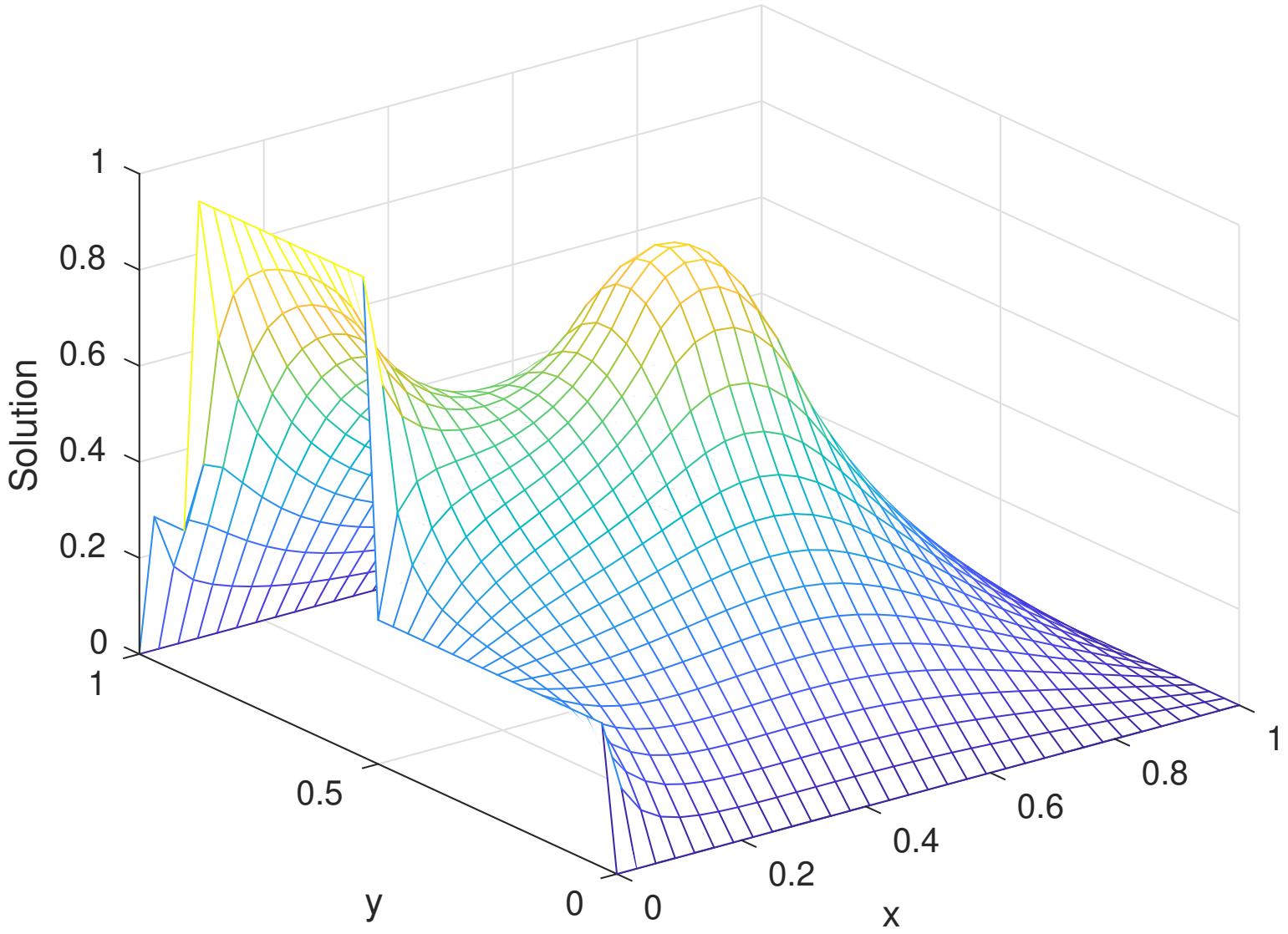
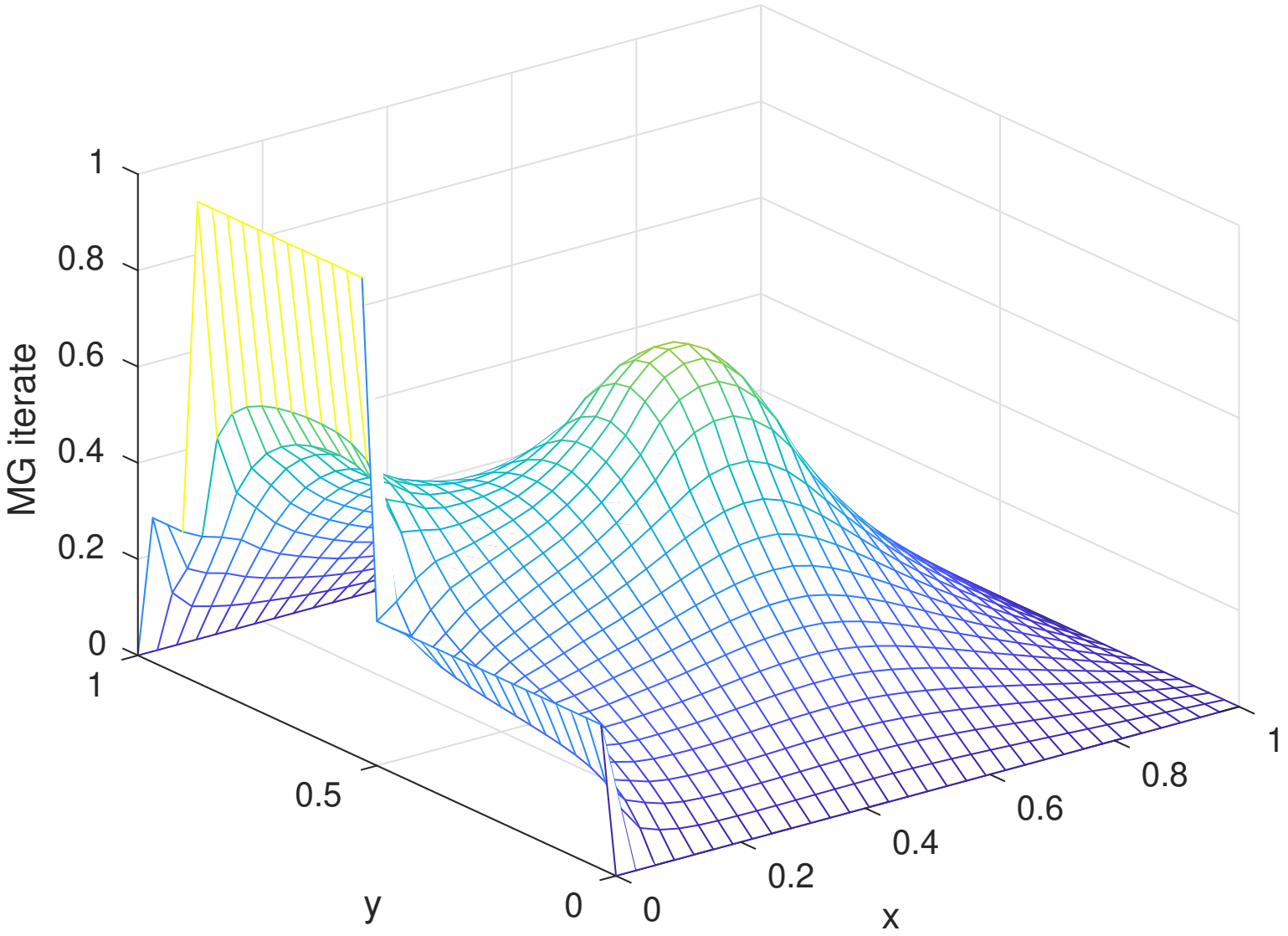
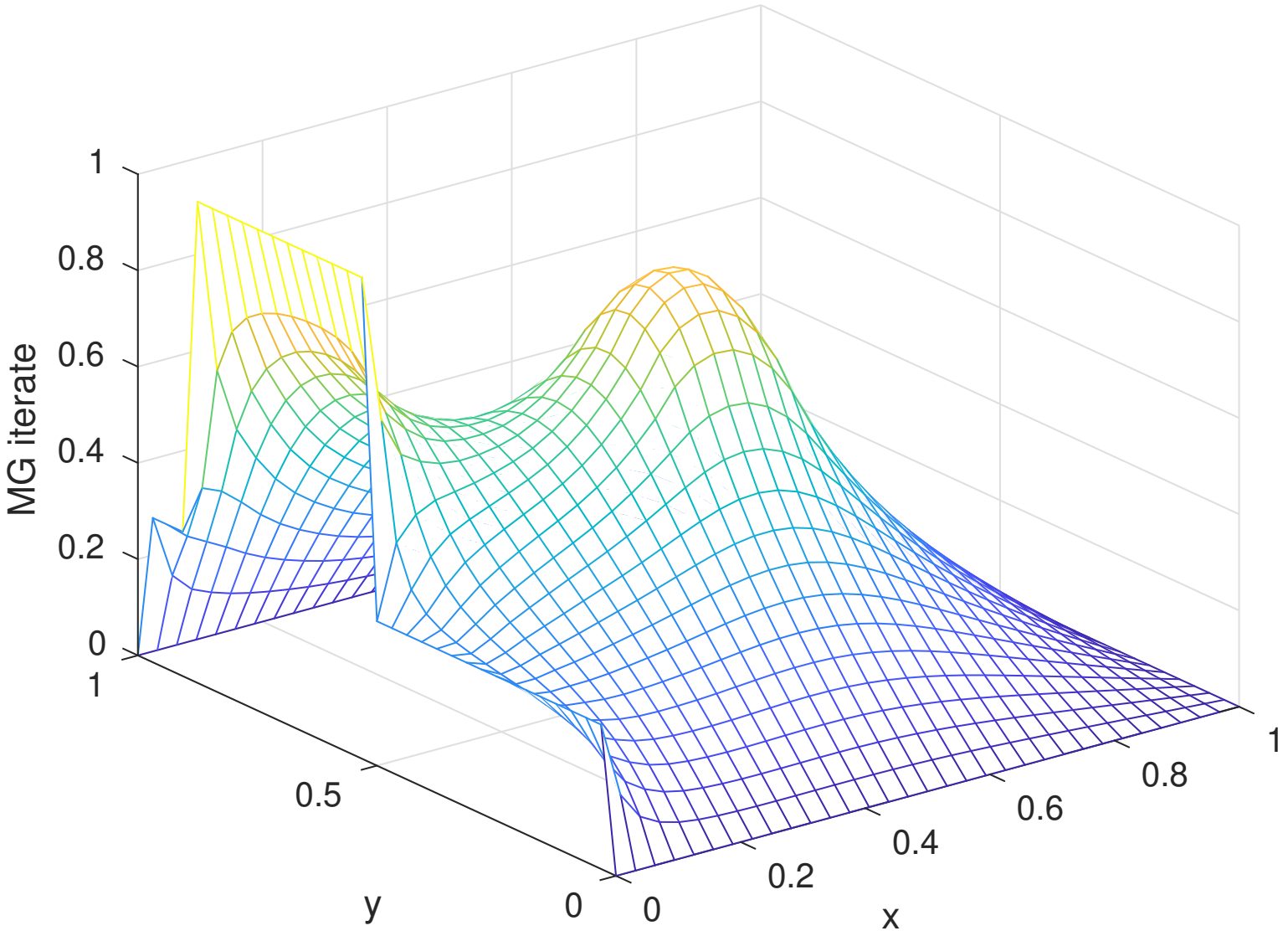
Example: temperature in a room



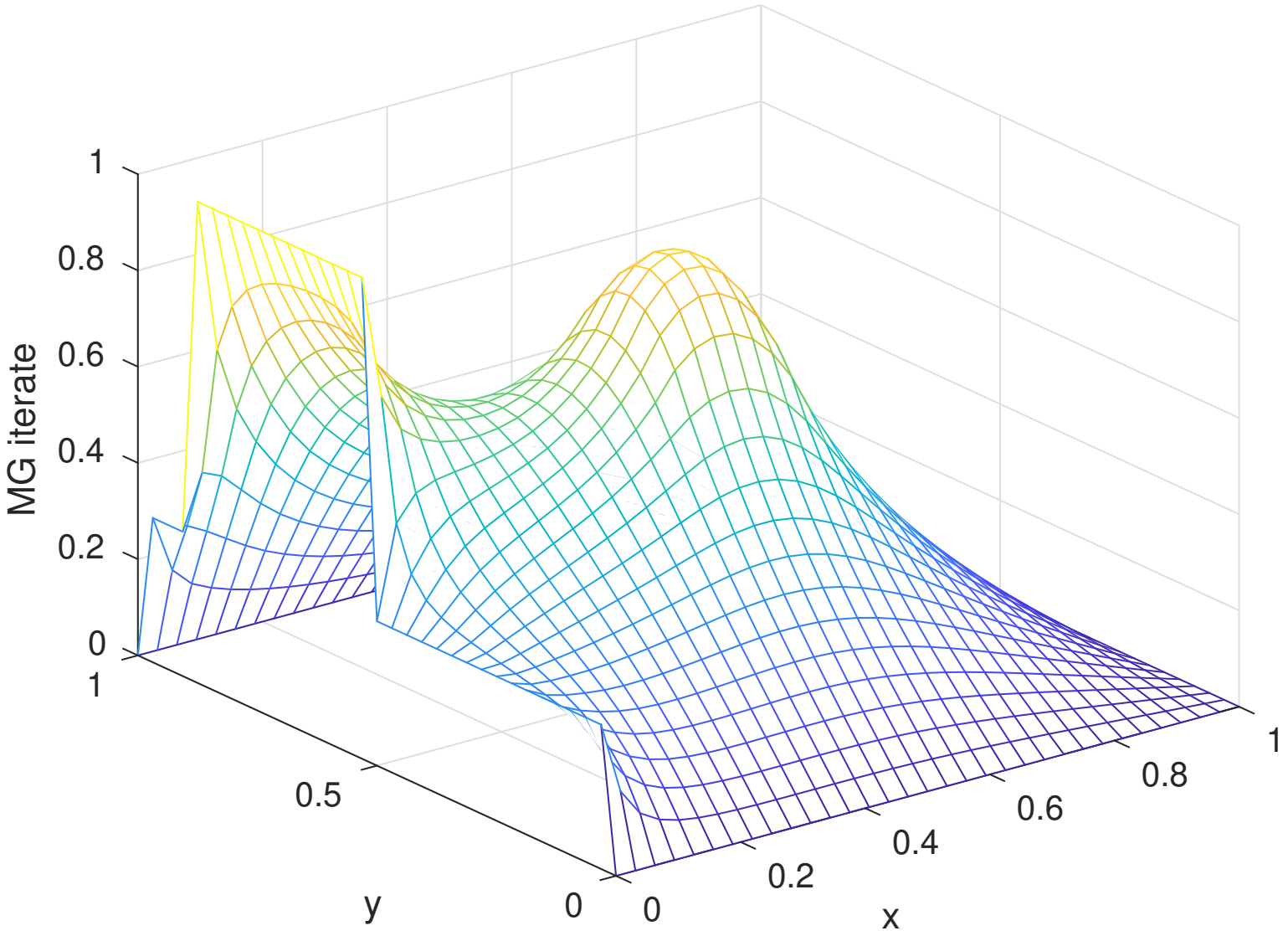
MG iteration 1



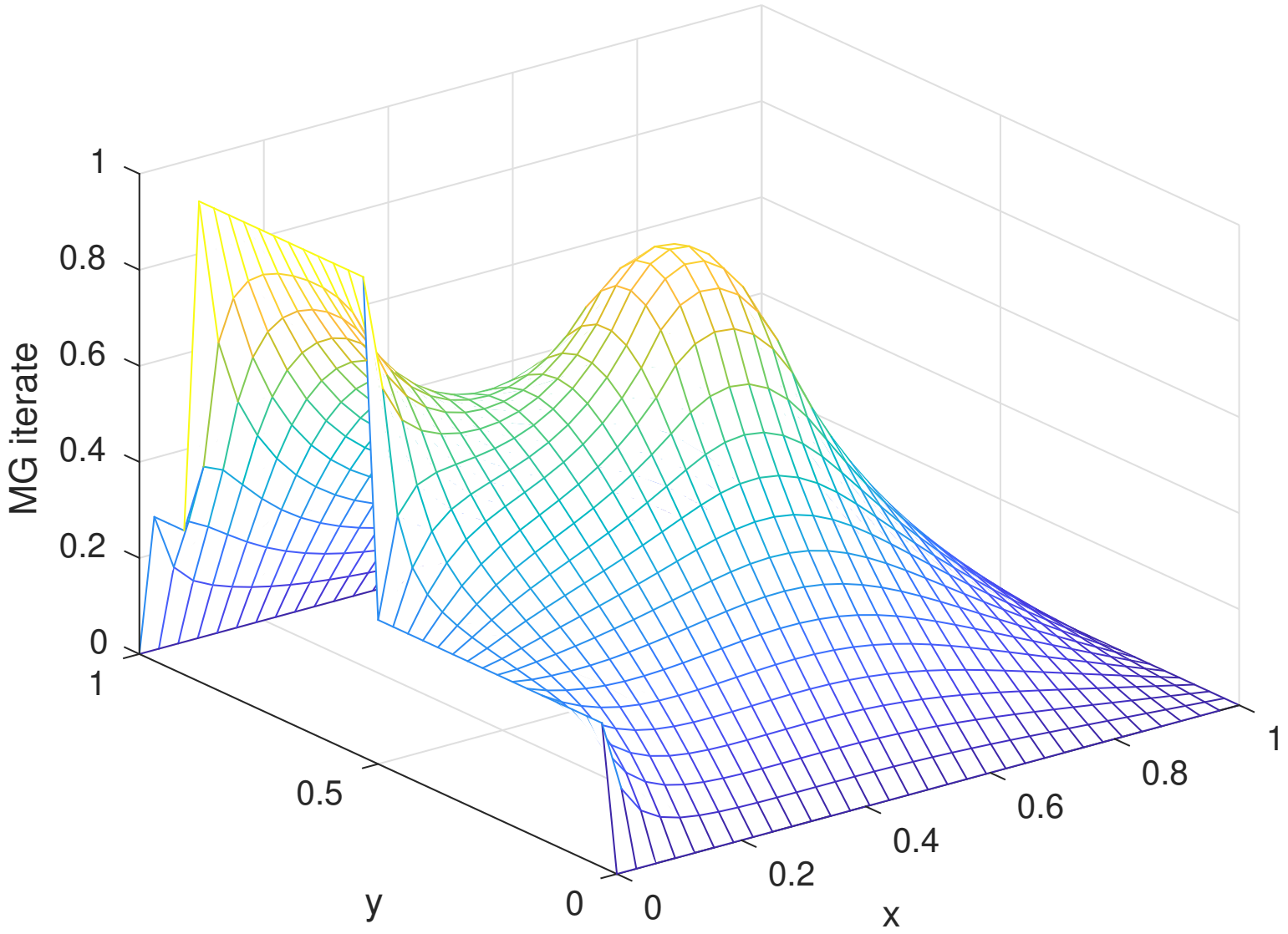
MG iteration 2



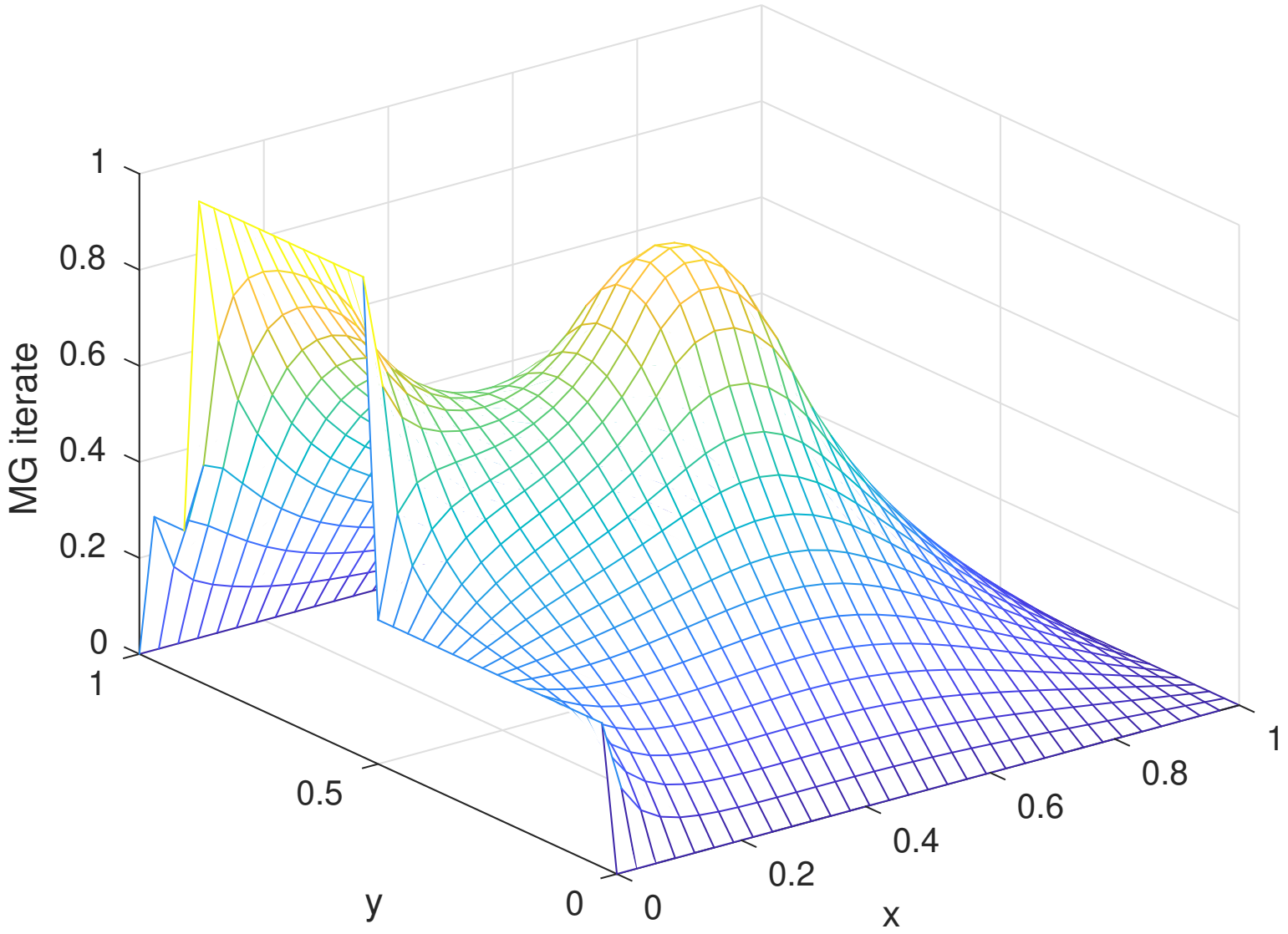
MG iteration 3



MG iteration 4



MG iteration 5



Seminal Contributions to Multigrid

Fedorenko (1961): A Relaxation Method for Solving Elliptic Difference Equations

Brandt 1972: Multi-Level Adaptive Technique (MLAT) for Fast Numerical Solution to Boundary Value Problems

Nicolaides 1975: On Multiple Grid and Related Techniques for Solving Discrete Elliptic Systems

Hackbusch 1976: A fast iterative method for solving Poisson's equation in a general region

Griebel (PhD thesis 1990): 23 floating point operations per grid point and V-cycle!

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Combination Means Preconditioning!

We have seen two types of methods:

1. Stationary iterative methods (Jacobi, Gauss-Seidel, SOR, Domain Decomposition and Multigrid).
2. Non-stationary iterative methods (Extrapolation, Chebyshev semi-iterative method, Krylov methods).

All stationary methods are of the form

$$\boldsymbol{u}^{n+1} = \boldsymbol{u}^n + M^{-1}(\boldsymbol{f} - A\boldsymbol{u}^n).$$

At the fixed point, we have the system

$$\boldsymbol{u} = \boldsymbol{u} + M^{-1}(\boldsymbol{f} - A\boldsymbol{u}) \quad \Longleftrightarrow M^{-1}A\boldsymbol{u} = M^{-1}\boldsymbol{f}.$$

Applying a Krylov method or extrapolation to solve this system, gives a much better residual polynomial than the stationary iteration which gives $p_n(M^{-1}A) = (I - M^{-1}A)^n$.

Always use Krylov acceleration!

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