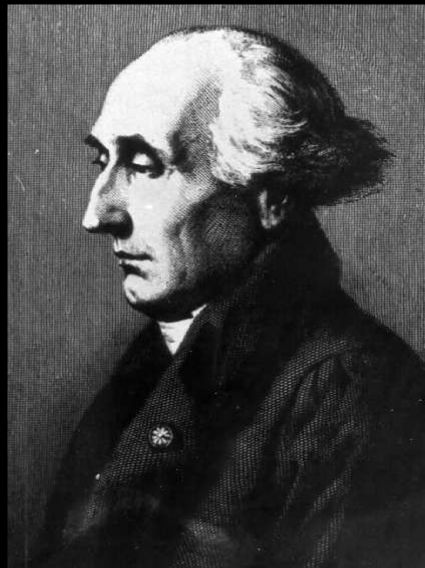
Buergi 1584



Lagrange 1780

Bernoulli 1742

Euler 1764/78

Zigzag Iterations

Joint work with

Philippe Henry

Fourier 1807

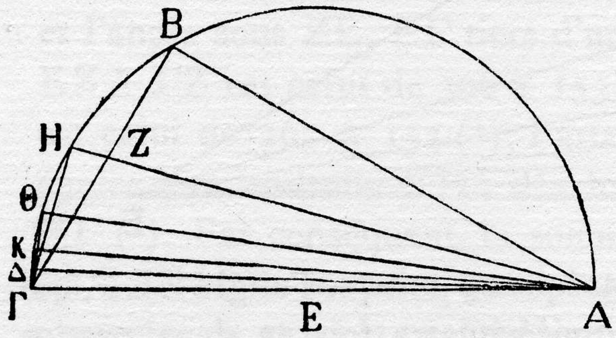
4000 years of scientific calculations...

Babylon 1850 B.C.

$$\sqrt{2} = 1,24\,51\,10$$

(one sqrt)

Archimedes 287–212 B.C.



d'où : $\frac{A\Gamma}{A\Gamma + AB} = \frac{\Gamma Z}{\Gamma Z + ZB}$, ou $\frac{A\Gamma}{\Gamma Z} = \frac{A\Gamma + AB}{B\Gamma}$, d'où : $\frac{AH}{H\Gamma} = \frac{A\Gamma + AB}{B\Gamma} = \frac{A\Gamma}{B\Gamma} + \frac{AB}{B\Gamma}$. Or, $\frac{A\Gamma}{B\Gamma} = \frac{1560}{780}$, et $\frac{AB}{B\Gamma} < \frac{1351}{780}$; donc $\frac{AH}{H\Gamma} < \frac{1560 + 1351}{780}$, ou $\frac{AH}{H\Gamma} < \frac{2911}{780}$. D'autre part, $\frac{\overline{AH}^2}{\overline{H\Gamma}^2} < \frac{(2911)^2}{(780)^2}$, d'où : $\frac{\overline{AH}^2 + \overline{H\Gamma}^2}{\overline{H\Gamma}^2} < \frac{2911^2 + 780^2}{780^2}$, ou $\frac{\overline{A\Gamma}^2}{\overline{H\Gamma}^2} < \frac{9082321}{608400}$, d'où, comme le texte : $\frac{A\Gamma}{H\Gamma} < \frac{3013\frac{3}{4}}{780}$.

1. On aura, comme dans le cas précédent : $\frac{A\Theta}{\Theta\Gamma} = \frac{A\Gamma + AH}{H\Gamma} = \frac{A\Gamma}{HA} + \frac{AH}{H\Gamma}$, d'où, substituant les valeurs de ces deux derniers termes : $\frac{A\Theta}{\Theta\Gamma} < \frac{3013\frac{3}{4} + 2911}{780}$, ou $\frac{A\Theta}{\Theta\Gamma} < \frac{5924\frac{3}{4}}{780}$, ou $\frac{A\Theta}{\Theta\Gamma} < \frac{\frac{4}{13} \times 5924\frac{3}{4}}{\frac{4}{13} \times 780}$, ou $\frac{A\Theta}{\Theta\Gamma} < \frac{1823}{240}$. D'autre part, $\frac{\overline{A\Theta}^2}{\overline{\Theta\Gamma}^2} < \frac{1823^2}{240^2}$, d'où $\frac{\overline{A\Theta}^2 + \overline{\Theta\Gamma}^2}{\overline{\Theta\Gamma}^2} < \frac{1823^2 + 240^2}{240^2}$, ou $\frac{\overline{A\Gamma}^2}{\overline{\Theta\Gamma}^2} < \frac{3380929}{57600}$, d'où, comme le texte : $\frac{A\Gamma}{\Theta\Gamma} < \frac{1838\frac{9}{11}}{240}$.

2. On aura de même : $\frac{AK}{K\Gamma} = \frac{A\Gamma + A\Theta}{\Theta\Gamma} = \frac{A\Gamma}{\Theta\Gamma} + \frac{A\Theta}{\Theta\Gamma}$, et, par substitution des valeurs trouvées pour ces deux derniers termes, il vient : $\frac{AK}{K\Gamma} < \frac{1838\frac{9}{11} + 1823}{240}$, ou $\frac{AK}{K\Gamma} < \frac{3661\frac{9}{11}}{240}$, ou $\frac{AK}{K\Gamma} < \frac{\frac{11}{40}3661\frac{9}{11}}{\frac{11}{40}240}$, ou $\frac{AK}{K\Gamma} < \frac{1007}{66}$. D'autre part, $\frac{\overline{AK}^2}{\overline{K\Gamma}^2} < \frac{1007^2}{66^2}$, d'où : $\frac{\overline{AK}^2 + \overline{K\Gamma}^2}{\overline{K\Gamma}^2} < \frac{1007^2 + 66^2}{66^2}$, ou $\frac{\overline{A\Gamma}^2}{\overline{K\Gamma}^2} < \frac{1018405}{14356}$, d'où, comme le texte : $\frac{A\Gamma}{K\Gamma} < \frac{1009\frac{1}{6}}{66}$.

3. On aura de même: $\frac{A\Lambda}{\Lambda\Gamma} = \frac{A\Gamma + AK}{K\Gamma} = \frac{A\Gamma}{K\Gamma} + \frac{AK}{K\Gamma}$, et, par substitution des valeurs précédentes : $\frac{A\Lambda}{\Lambda\Gamma} < \frac{1009\frac{1}{6}}{66} + \frac{1007}{66}$, ou $\frac{A\Lambda}{\Lambda\Gamma} < \frac{2016\frac{1}{6}}{66}$. D'autre part, $\frac{\overline{A\Lambda}^2 + \overline{\Lambda\Gamma}^2}{\overline{\Lambda\Gamma}^2} < \frac{(2016\frac{1}{6})^2 + 66^2}{66^2}$, ou $\frac{\overline{A\Gamma}^2}{\overline{\Lambda\Gamma}^2} < \frac{4069284\frac{1}{36}}{4356}$, d'où, comme le texte: $\frac{A\Gamma}{\Lambda\Gamma} < \frac{2017\frac{1}{4}}{66}$.

1. Sous-entendu : περίμετρος, le périmètre (du cercle).

2. La relation de la note avant-précédente donne, par inversion : $\frac{\Lambda\Gamma}{A\Gamma} > \frac{66}{2017\frac{1}{4}}$, d'où, observant que $96 \times \Lambda\Gamma =$ périmètre polygone inscrit de 96 côtés : $\frac{\text{périmètre polygone de 96 côtés}}{\text{diamètre cercle}} > \frac{96 \times 66}{2017\frac{1}{4}}$, ou $> \frac{6336}{2017\frac{1}{4}}$. Or, $\frac{6336}{2017\frac{1}{4}} > 3\frac{10}{71}$, d'où périmètre polygone de 96 côtés $> 3\frac{10}{71}$ diamètre cercle, d'où, à fortiori, suivant le texte : Circonférence cercle $> 3\frac{10}{71}$ diamètre.

$$3\frac{10}{71} < \pi < 3\frac{1}{7}$$

(using 8 sqrts)

Ptolemy ≈150 A.D.

ΚΑΝΟΝΙΟΝ ΤΩΝ ΕΝ ΚΥΚΛῼ ΕΥΘΕΙΩΝ.

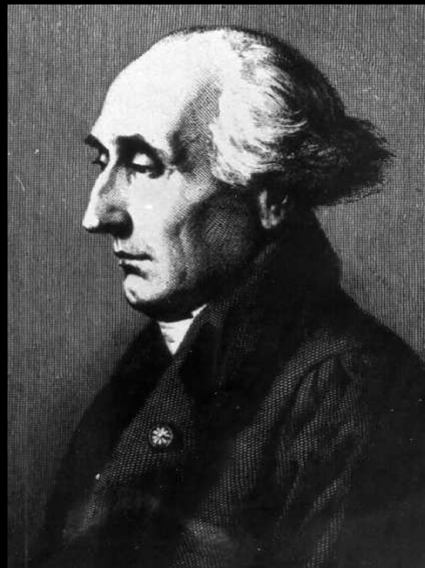
ΠΕΡΙΦΕΡΕΙΩΝ. Μοιρῶν.		ΕΥΘΕΙΩΝ. Μ.	Π.	Δ.	ΕΞΗΚΟΣΤΩΝ. Μ.	Π.	Δ.	Τ.
ο̄	ς″	ο̄	λα	κε	ο̄	α	β	ν
α	ο̄	α	β	ν	ο̄	α	β	ν
α	ς″	α	λδ	ιε	ο̄	α	β	ν
β	ο̄	β	ε	μ	ο̄	α	β	ν
β	ς″	β	λζ	δ	ο̄	α	β	μη
γ	ο̄	γ	η	κη	ο̄	α	β	μη
γ	ς″	γ	λθ	νβ	ο̄	α	β	μη
δ	ο̄	δ	ια	ις	ο̄	α	β	μζ
δ	ς″	δ	μβ	μ	ο̄	α	β	μζ
ε	ο̄	ε	ιδ	δ	ο̄	α	β	μς
ε	ς″	ε	με	κζ	ο̄	α	β	με
ς	ο̄	ς	ις	μθ	ο̄	α	β	μδ
ς	ς″	ς	μη	ια	ο̄	α	β	μγ
ζ	ο̄	ζ	ιθ	λγ	ο̄	α	β	μβ
ζ	ς″	ζ	ν	νδ	ο̄	α	β	μα
η	ο̄	η	κβ	ιε	ο̄	α	β	μ
η	ς″	η	νγ	λε	ο̄	α	β	λθ
θ	ο̄	θ	κδ	νδ	ο̄	α	β	λη

Table of chords of the circle

⇒ Regiomontanus 1533

⇒ Kepler 1609, Newton 1687

Buergi 1584



Lagrange 1780

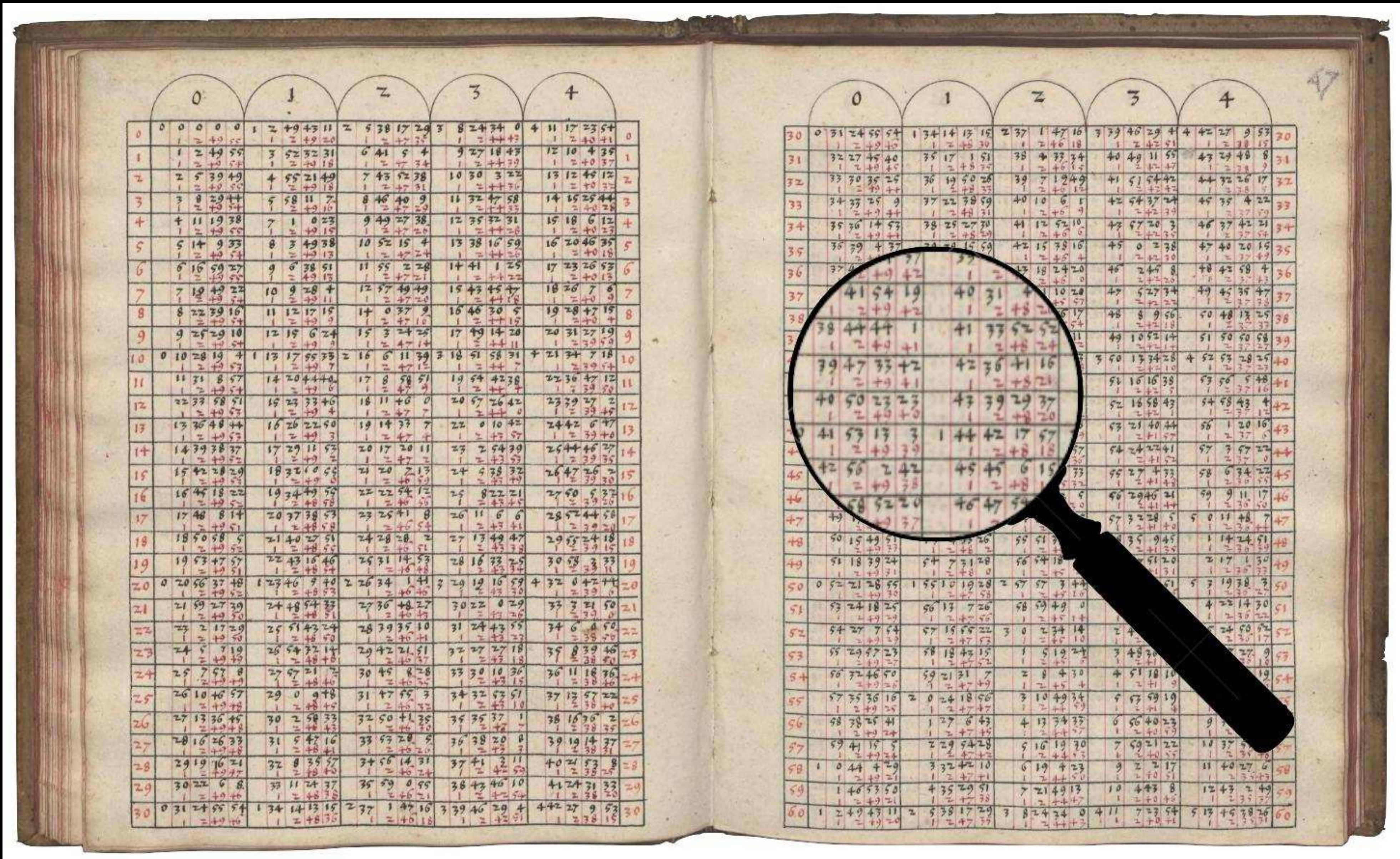
Bernoulli 1742

Euler 1764/78

1. Jost Bürgi's *Fundamentum Astronomiæ*

Fourier 1807

Buergi: 36 pages like this!



(http://www.bibliotekacyfrowa.pl/dlibra)

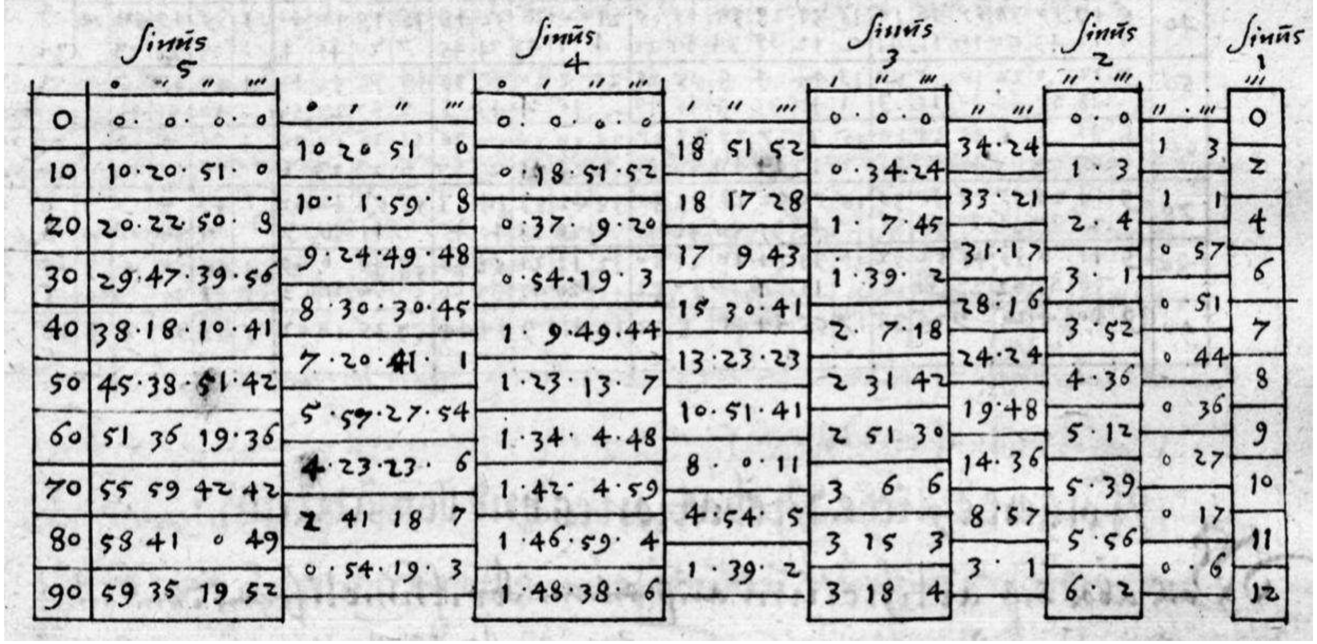
Discovered by Menso Folkerts in 2013 in Univ. Library of Wrocław

Buergi's method explained in this EXEMPLUM

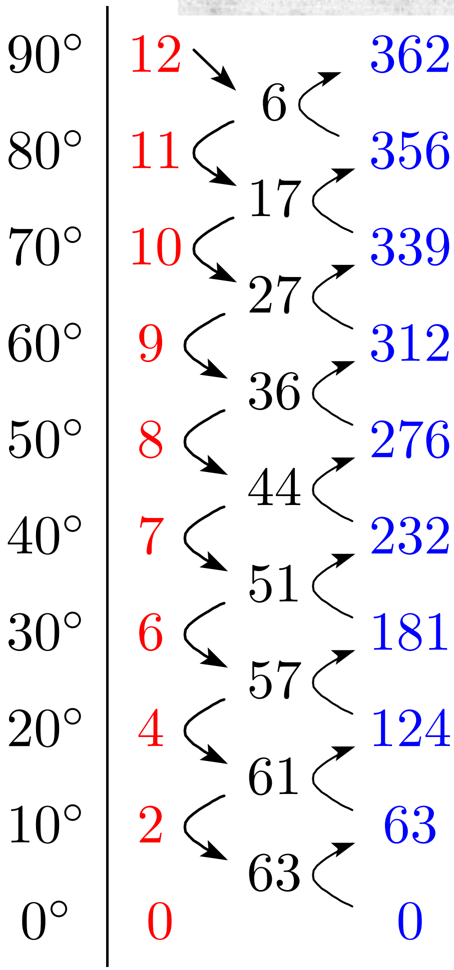
	Sinus 5 (° ′ ″ ‴)	(° ′ ″ ‴)	Sinus 4 (° ′ ″ ‴)	(′ ″ ‴)	Sinus 3 (′ ″ ‴)	(″ ‴)	Sinus 2 (″ ‴)	(″ ‴)	Sinus 1 (‴)
0	0 · 0 · 0 · 0	10 20 51 0	0 · 0 · 0 · 0	18 51 52	0 · 0 · 0	34 · 24	0 · 0	1 3	0
10	10 · 20 · 51 · 0	10 · 1 · 59 · 8	0 · 18 · 51 · 52	18 17 28	0 · 34 · 24	33 · 21	1 · 3	1 1	2
20	20 · 22 · 50 · 9	9 · 24 · 49 · 48	0 · 37 · 9 · 20	17 9 43	1 · 7 · 45	31 · 17	2 · 4	0 57	4
30	29 · 47 · 39 · 56	8 · 30 · 30 · 45	0 · 54 · 19 · 3	15 30 · 41	1 · 39 · 2	28 · 16	3 · 1	0 51	6
40	38 · 18 · 10 · 41	7 · 20 · 41 · 1	1 · 9 · 49 · 44	13 · 23 · 23	2 · 7 · 18	24 · 24	3 · 52	0 44	7
50	45 · 38 · 51 · 42	5 · 59 · 27 · 54	1 · 23 · 13 · 7	10 · 51 · 41	2 31 42	19 · 48	4 · 36	0 36	8
60	51 36 19 · 36	4 · 23 · 23 · 6	1 · 34 · 4 · 48	8 · 0 · 11	2 51 30	14 · 36	5 · 12	0 27	9
70	55 59 42 42	2 41 18 7	1 · 42 · 4 · 59	4 · 54 · 5	3 6 6	8 57	5 · 39	0 17	10
80	58 41 0 49	0 · 54 · 19 · 3	1 · 46 · 59 · 4	1 · 39 · 2	3 15 3	3 · 1	5 · 56	0 · 6	11
90	59 35 19 52		1 · 48 · 38 · 6		3 18 4		6 · 2		12

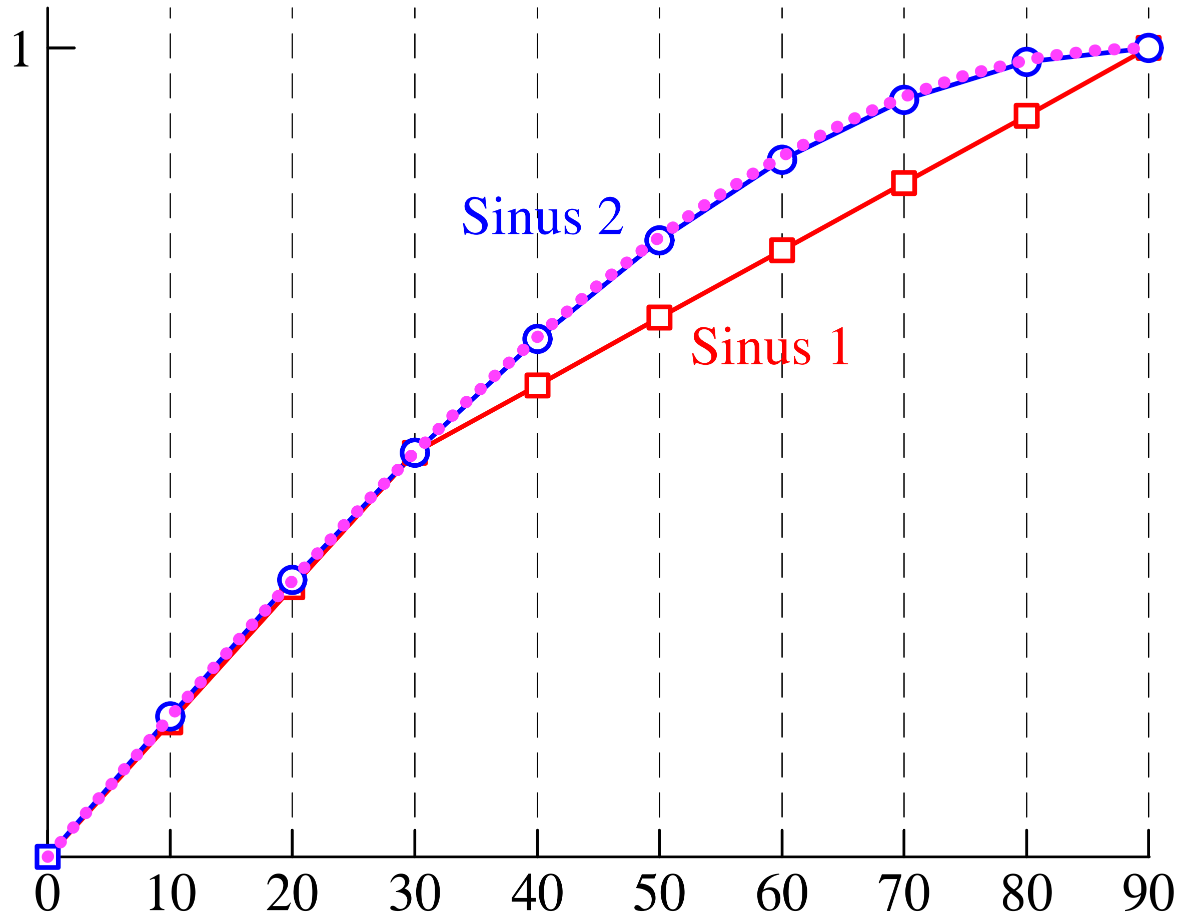
Jost Bürgi 1584 (http://www.bibliotekacyfrowa.pl/dlibra)

Under-
stand?



	Sinus 5		Sinus 4		Sinus 3		Sinus 2		Sinus 1
	° ′ ″ ‴	° ′ ″ ‴	° ′ ″ ‴	′ ″ ‴	′ ″ ‴	″ ‴	″ ‴	″ ‴	‴
0	0. 0. 0. 0		0. 0. 0. 0		0. 0. 0		0. 0		0
10	10.20.51.0	10 20 51 0	0.18.51.52	18 51 52	0.34.24	34.24	1.3	1 3	2
20	20.22.50.9	10 1.59.8	0.37.9.20	18 17 28	1.7.45	33.21	2.4	1 1	4
30	29.47.39.56	9.24.49.48	0.54.19.3	17 9 43	1.39.2	31.17	3.1	0 57	6
40	38.18.10.41	8 30.30.45	1.9.49.44	15 30.41	2.7.18	28.16	3.52	0 51	7
50	45.38.51.42	7.20.41.1	1.23.13.7	13.23.23	2 31 42	24.24	4.36	0 44	8
60	51 36 19.36	5.59.27.54	1.34.4.48	10.51.41	2 51 30	19.48	5.12	0 36	9
70	55 59 42 42	4.23.23.6	1.42.4.59	8.0.11	3 6 6	14.36	5.39	0 27	10
80	58 41 0 49	2 41 18 7	1.46.59.4	4.54.5	3 15 3	8.57	5.56	0 17	11
90	59 35 19 52	0.54.19.3	1.48.38.6	1.39.2	3 18 4	3.1	6.2	0.6	12



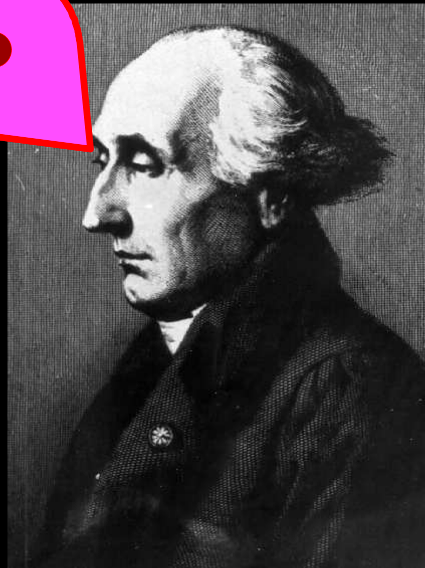


90°	12		362		11884		391086		12871192
		6		181		5942		195543	
80°	11		356		11703		385144		12675649
		17		537		17645		580687	
70°	10		339		11166		367499		12094962
		27		876		28811		948186	
60°	9		312		10290		338688		11146776
		36		1188		39101		1286874	
50°	8		276		9102		299587		9859902
		44		1464		48203		1586461	
40°	7		232		7638		251384		8273441
		51		1696		55841		1837845	
30°	6		181		5942		195543		6435596
		57		1877		61783		2033388	
20°	4		124		4065		133760		4402208
		61		2001		65848		2167148	
10°	2		63		2064		67912		2235060
		63		2064		67912		2235060	
0°	0		0		0		0		0

Normalized to $\sin 90^\circ = 1$ $\Rightarrow$ maximal error:

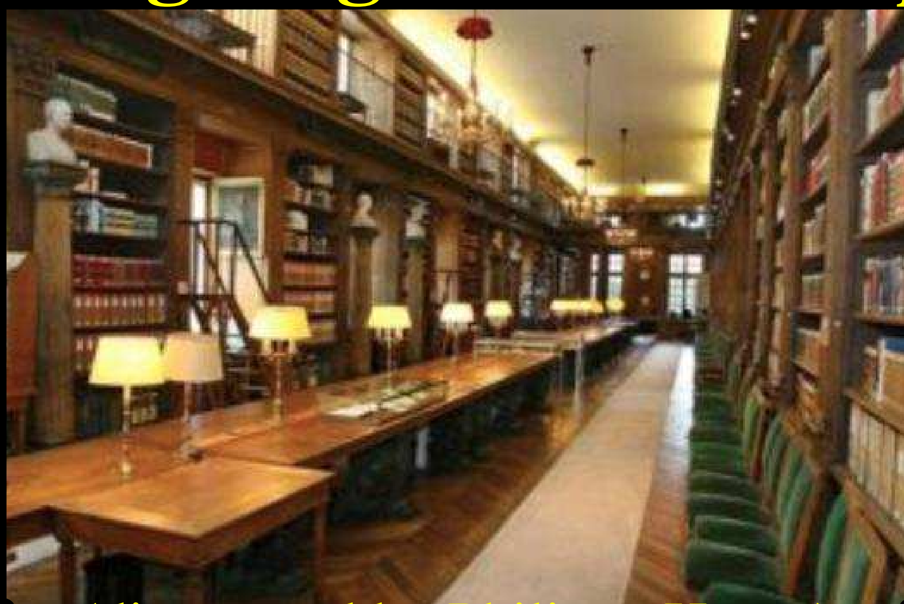
	Sinus 1	Sinus 2	Sinus 3	Sinus 4	Sinus 5
maxerr	0.11602540	0.00414695	0.00015533	0.00000617	0.00000025
ratio		27.978	26.698	25.195	24.423

$\mapsto$ presented to Rudolph II $\mapsto$ lost for 400 years.

Buergi 1584 Lagrange 1780 Bernoulli 1742 Euler 1764/78

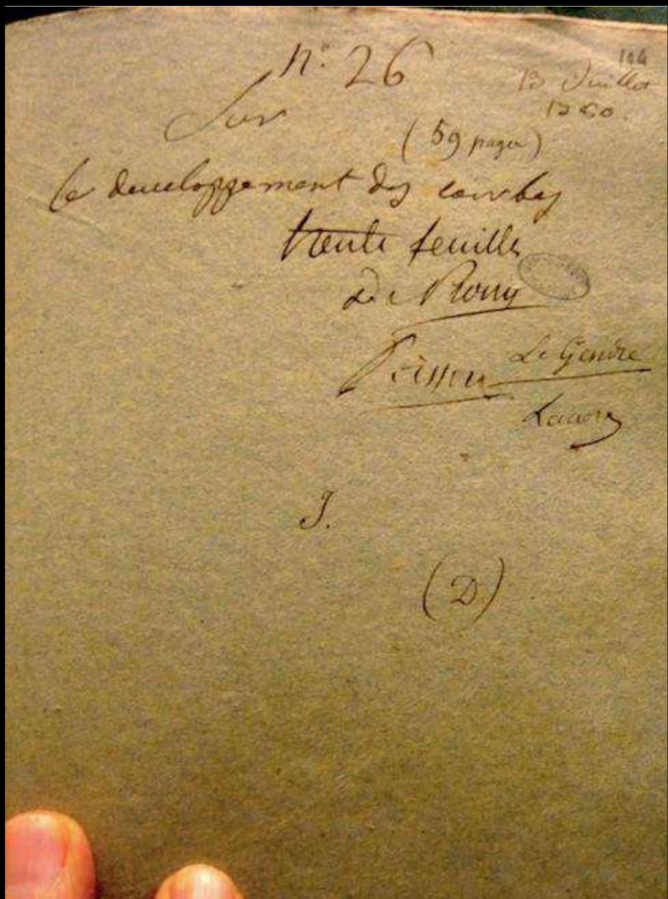
2. Lagrange's manuscript

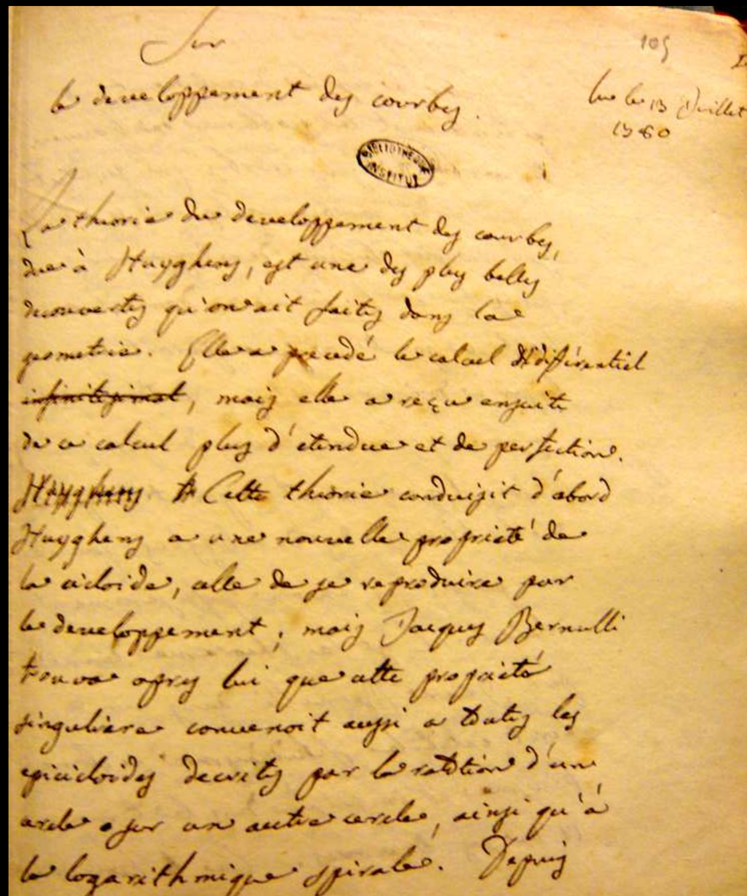


(discovered by Philippe Henry)

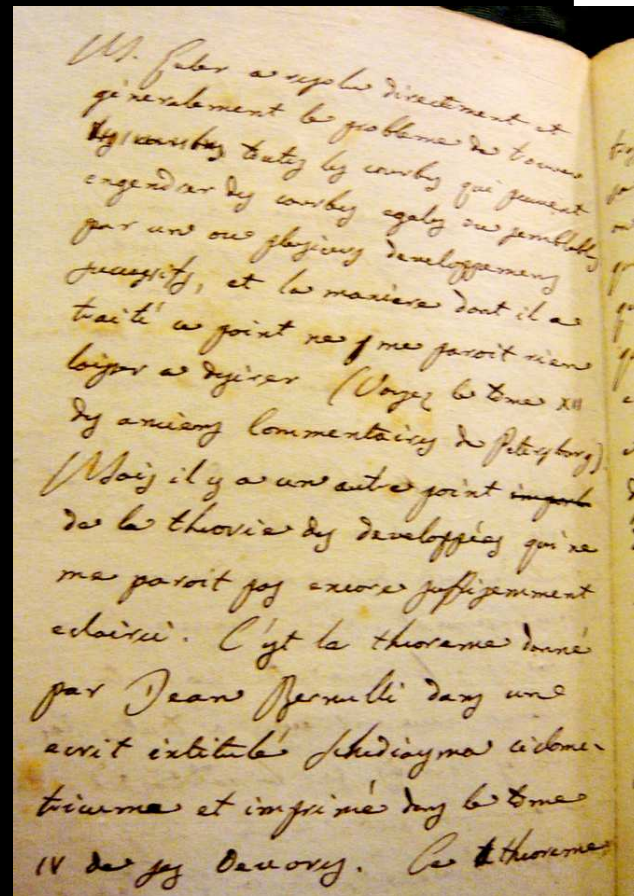
Fourier 1807

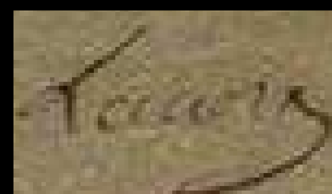
Lagrange's manuscript :
Le developpement des courbes (59 handwritten pages)

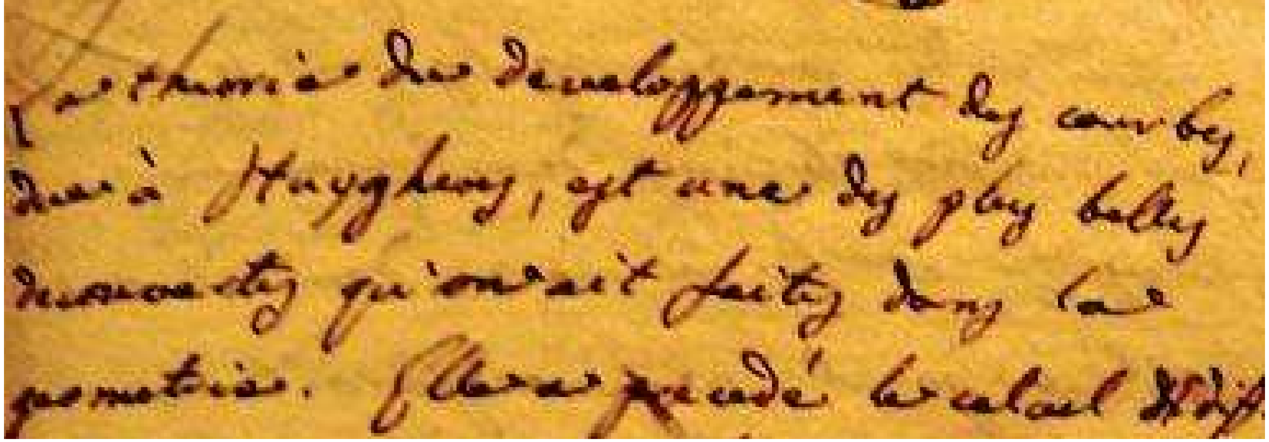

N°. 26
104
13 Juillet 1760.
Sur
(59 pages)
le developpement des courbes
trente feuilles
de Prony
Poisson
Le Gendre
Lacroix
J.
(2)


105
Sur
le developpement des courbes.
lu le 13 Juillet 1760

La theorie du developpement des courbes, due à Huyghens, est une des plus belles decouvertes qu'on ait faites dans la geometrie. Elle a precedé le calcul differentiel ~~infinitesimal~~, mais elle a reçu ensuite de ce calcul plus d'étendue et de perfection. ~~Huyghens~~ Cette theorie conduisit d'abord Huyghens à une nouvelle proprieté de la cicloide, celle de se reproduire par le developpement; mais Jacques Bernoulli trouva après lui que cette proprieté singuliere convenoit aussi à toutes les epicicloides decrites par la rotation d'un cercle sur un autre cercle, ainsi qu'à la logarithmique spirale. Depuis


M. Euler a resolu directement et generalement le probleme de trouver toutes les courbes qui peuvent engendrer des courbes egales ou semblables par un ou plusieurs developpemens successifs, et la maniere dont il a traité ce point ne me paroit rien laisser à desirer (Voyez le tome XII des anciens Commentaires de Petersbourg). Mais il y a un autre point ~~import~~ de la theorie des developpées qui ne me paroit pas encore suffisamment eclairci. C'est le theoreme donné par Jean Bernoulli dans un ecrit intitulé Schediasma (?) [illegible] et imprimé dans le tome IV de ses Oeuvres. Ce ~~theoreme~~

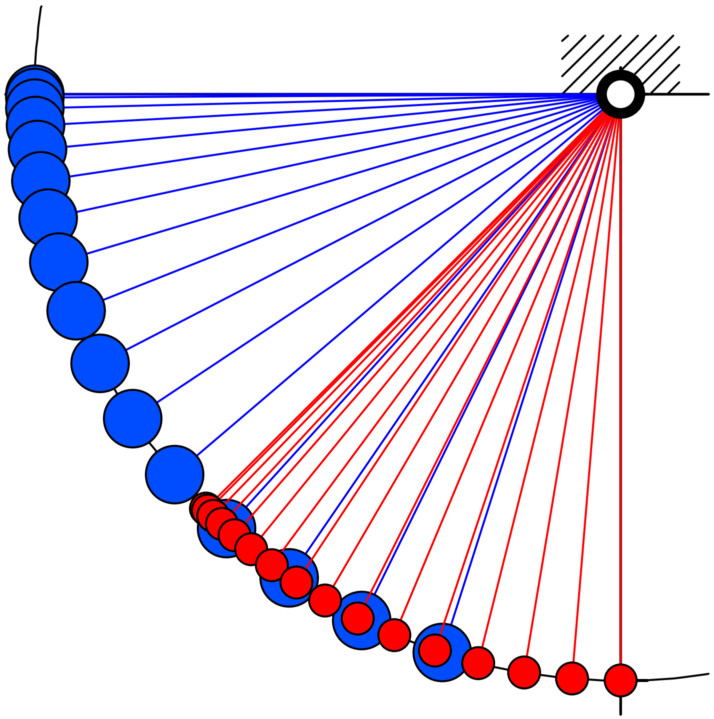


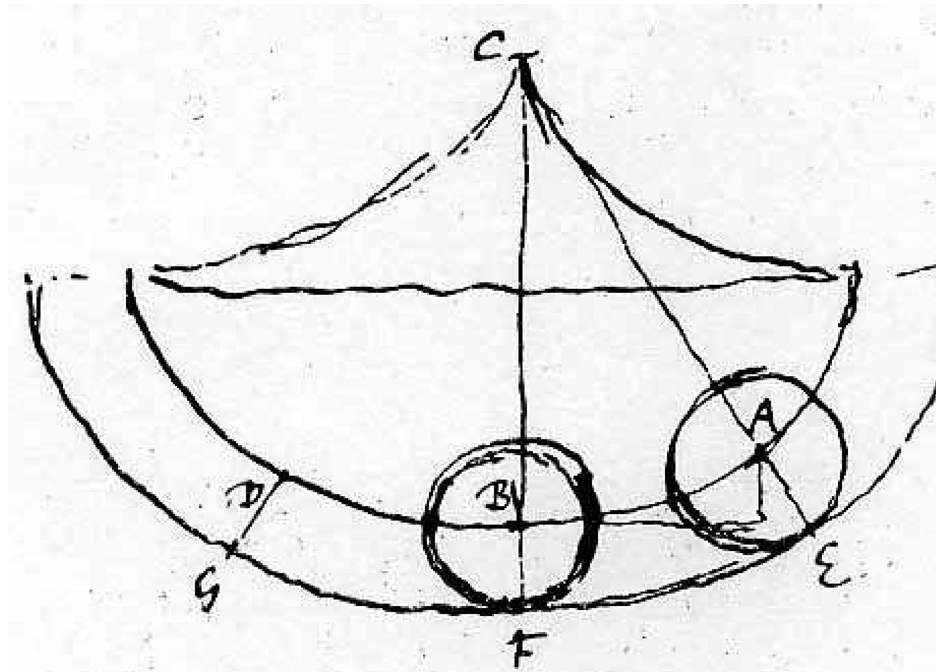


Lagrange:
La theorie du developpement
des courbes, due à Huyghens,
est une des plus belles decouvertes
qu'on ait faites dans la geometrie.

Christiaan Huygens (1629–1695):

M. Mersenne ⇒ Huygens: construct precise pendulum clocks!



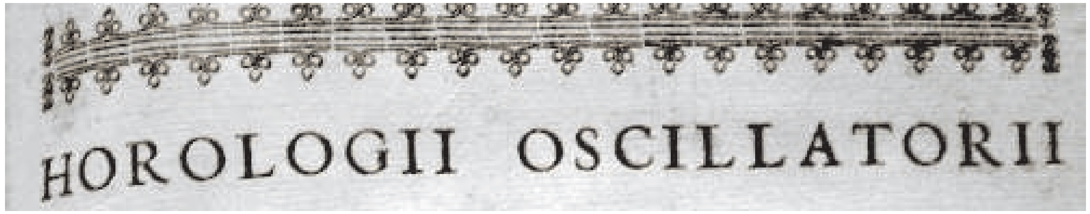


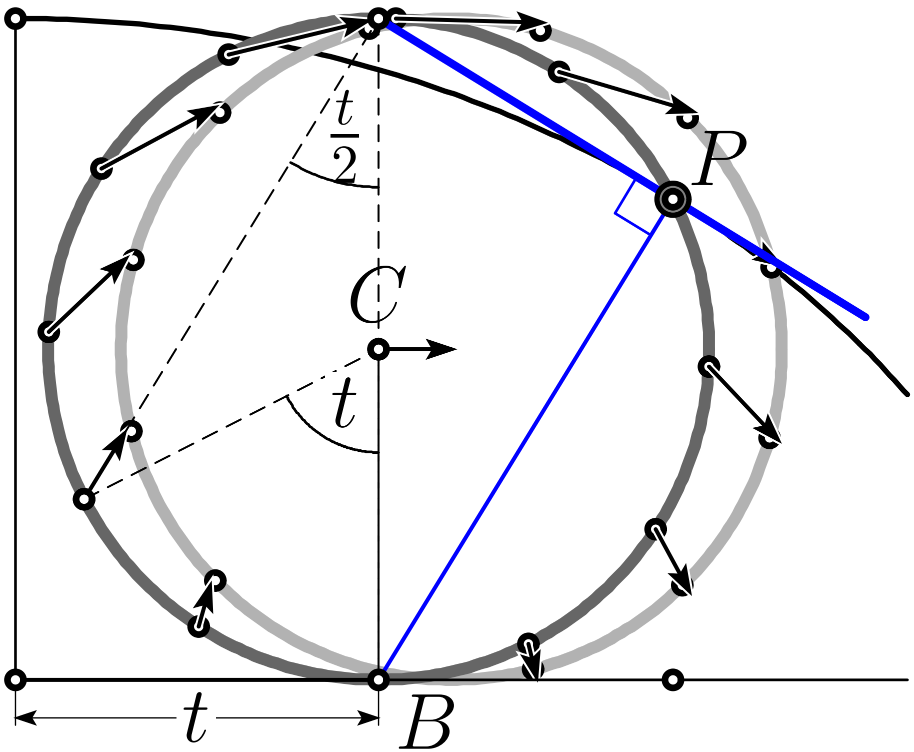
Problem: circul.pend. **not isochrone!**

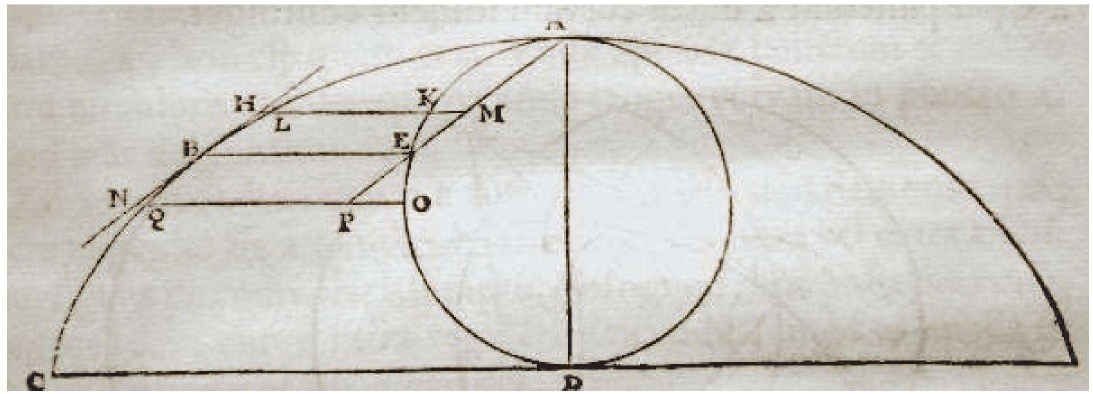
Idea: Must be steeper towards ends ⇒ **cycloid !!**

The Cycloid (Galilei 1599, Roberval 1640, Torricelli 1644, Pascal 1658 ("Roulette"), Huygens 1658, Joh. Bernoulli 1692)

(Huygens 1673):







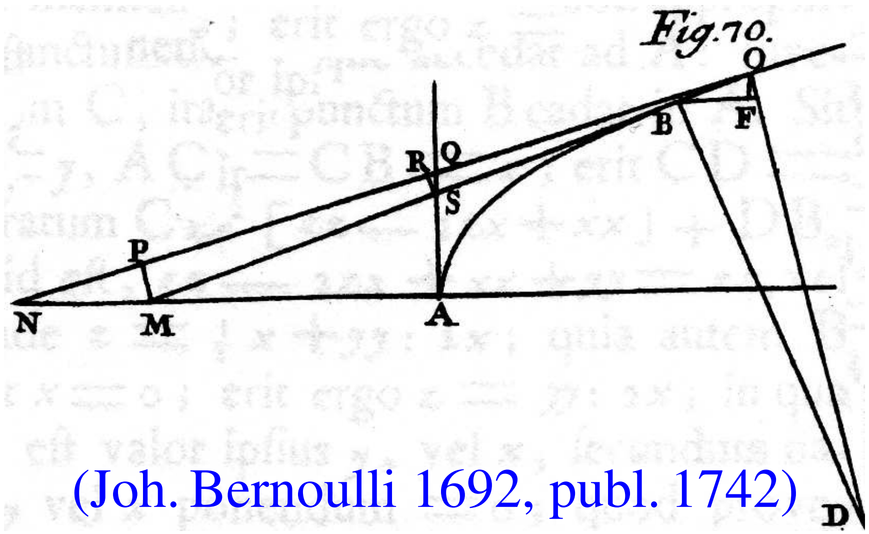
(Huygens 1673)

Thm. 1. Tangent in P is perpendicular to PB.

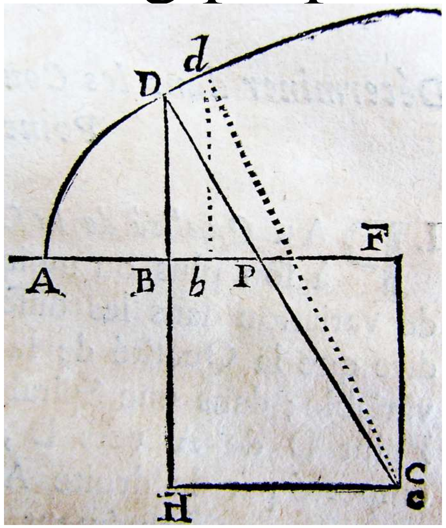
(because at any moment the circle rotates around the base point B, which remains fixed)

(Original proofs more complicated).

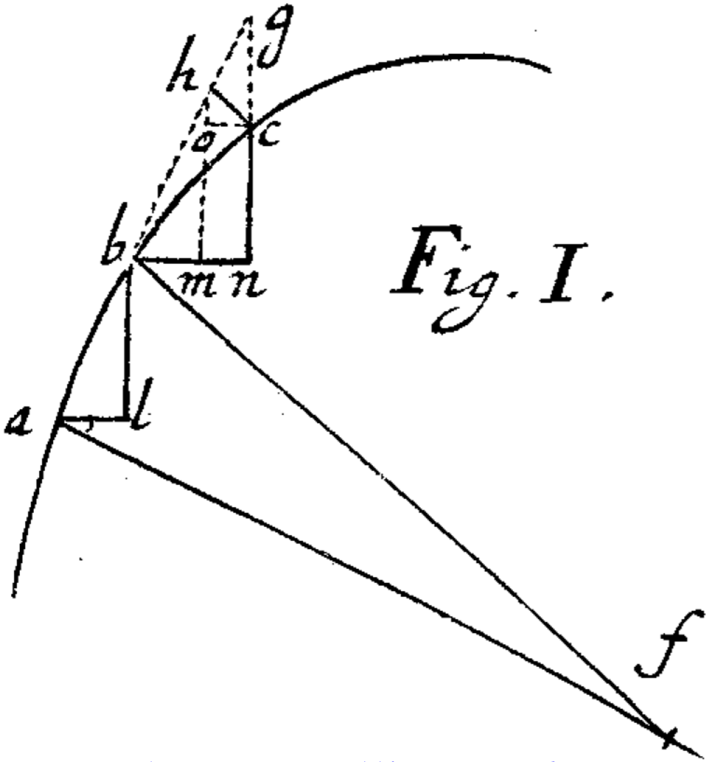
Center of curvature (= inters. neighbouring perpendiculars):



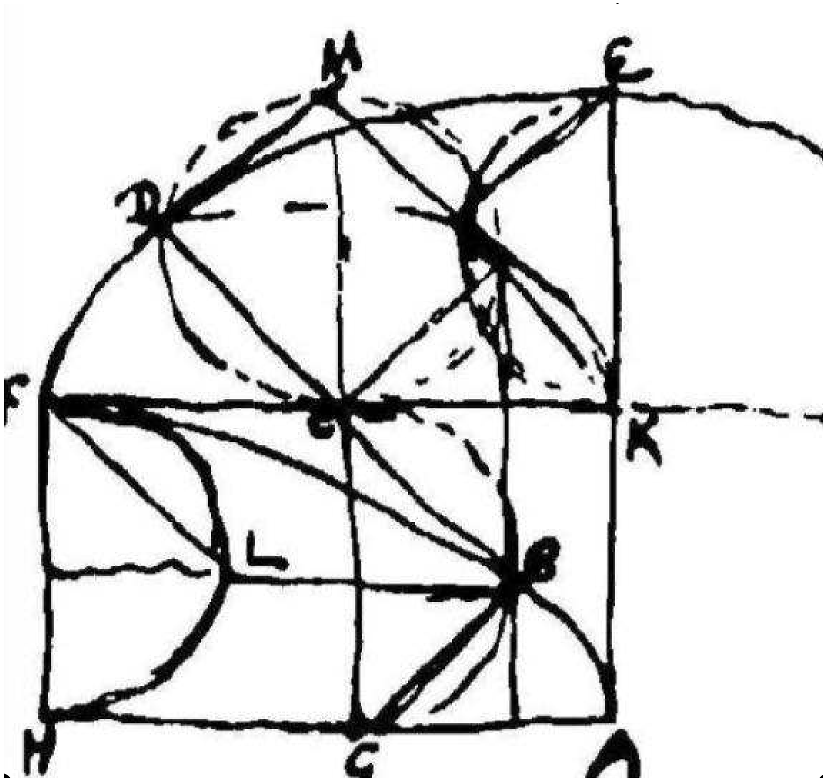
(Joh. Bernoulli 1692, publ. 1742)



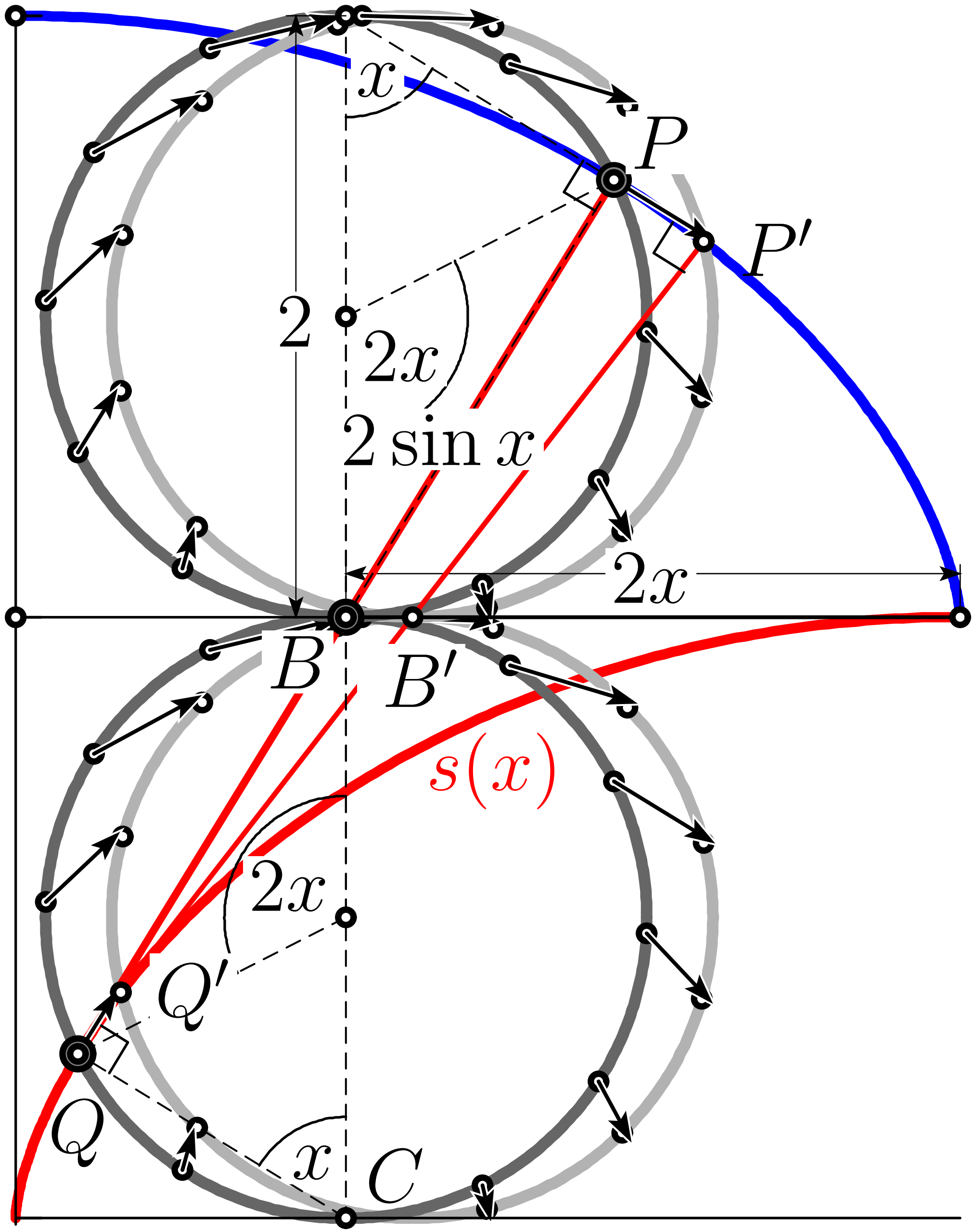
(Newton 1671, publ. 1736)



(Jak. Bernoulli AE 1691)



Idea: (Huygens 1659) add second circle



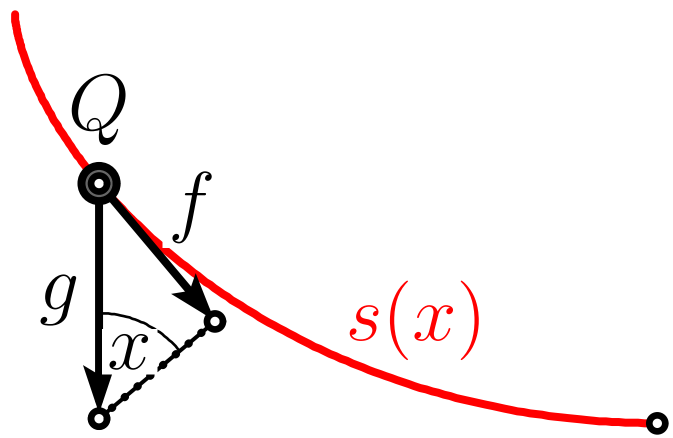
Thm. 2. Q (with $PB = BQ$) ist center of curvature in P

Thm. 3. Q on **evolute** (= **identical cycloid**)

P on **involute**.

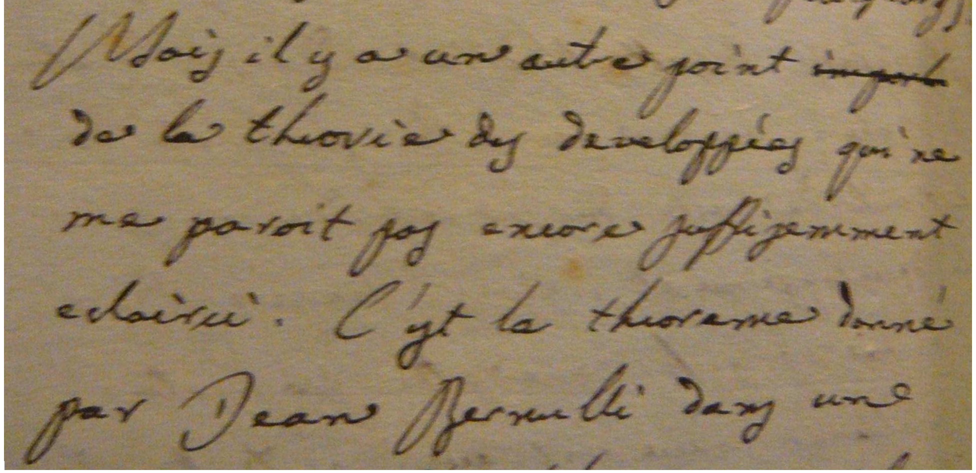
Thm. 4. Arc length

$s(x) = PQ = 2PB = 4\sin x$.



Thm. 5. Acc. force $f = g \sin x = \text{Const} \cdot s(x)$

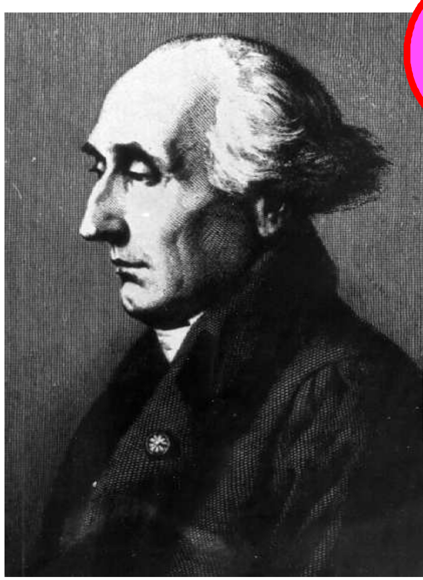
(harmonic oscillator) $\Rightarrow$ **isochronous pend.**


Mais il y a un autre point ~~import~~ de la theorie des developpées qui ne me paroit pas encore suffisemment eclairci. C'est le theoreme donné par Jean Bernoulli dans une

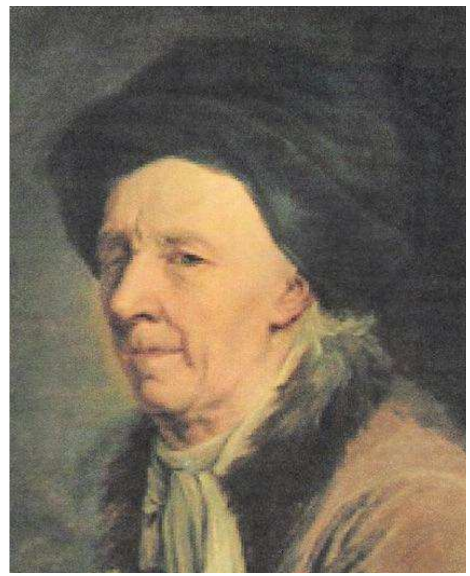
Lagrange:
Mais il y a un autre point
de la theorie des developpées
qui ne me paroit pas encore
suffisemment eclairci. C’est le
theoreme donné par Jean Bernoulli...

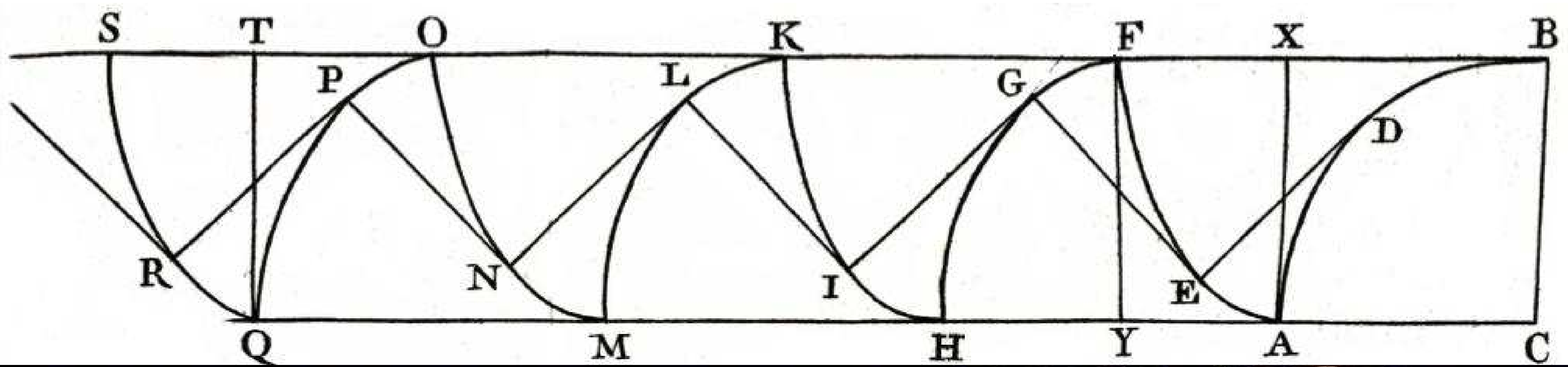
3. Johann Bernoulli’s
DE EVOLUTIONE SUCCESSIVA ET ALTERNANTE

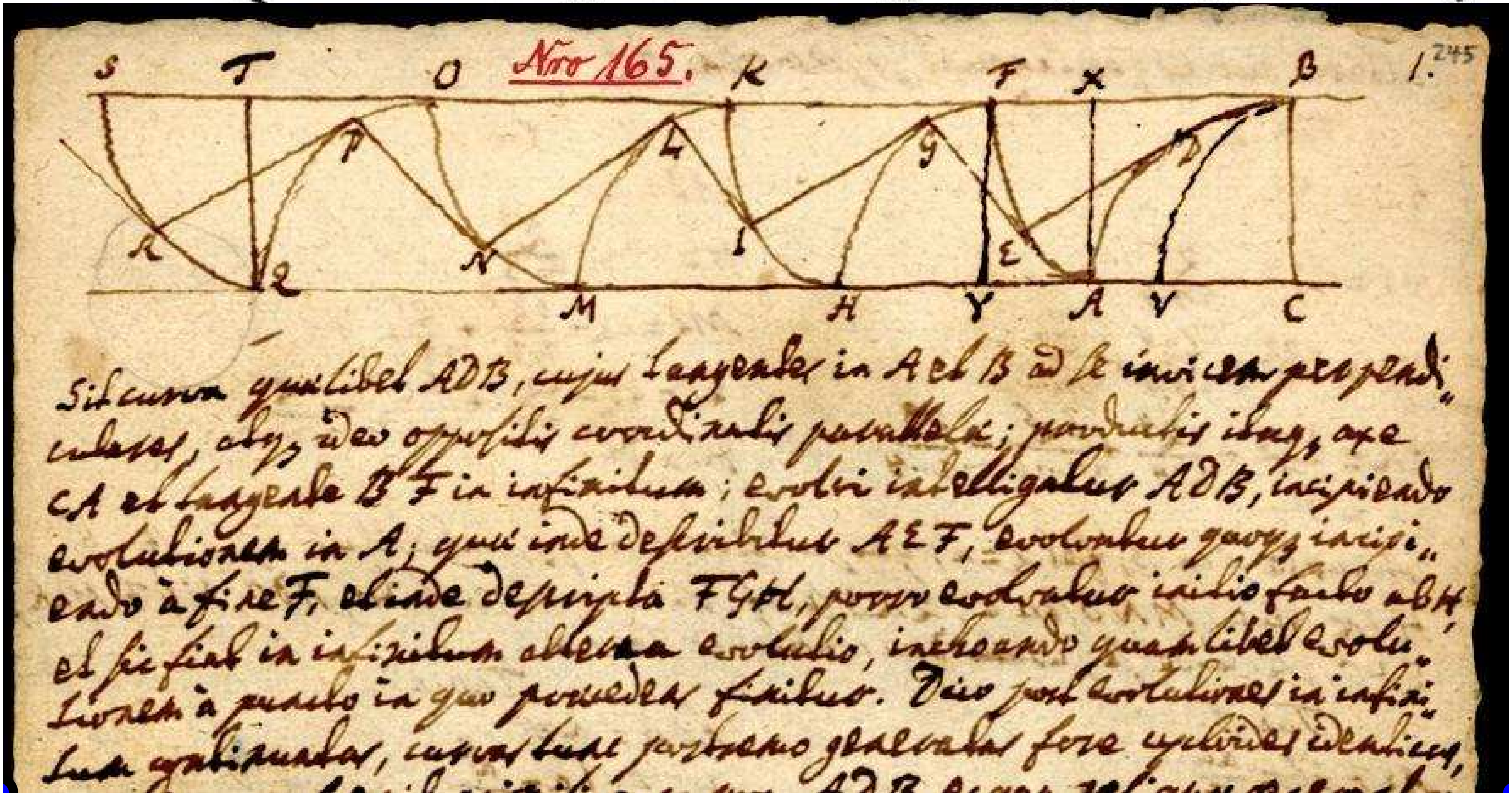

Buergi 1584


Lagrange 1780


Bernoulli 1742


Euler 1764/78



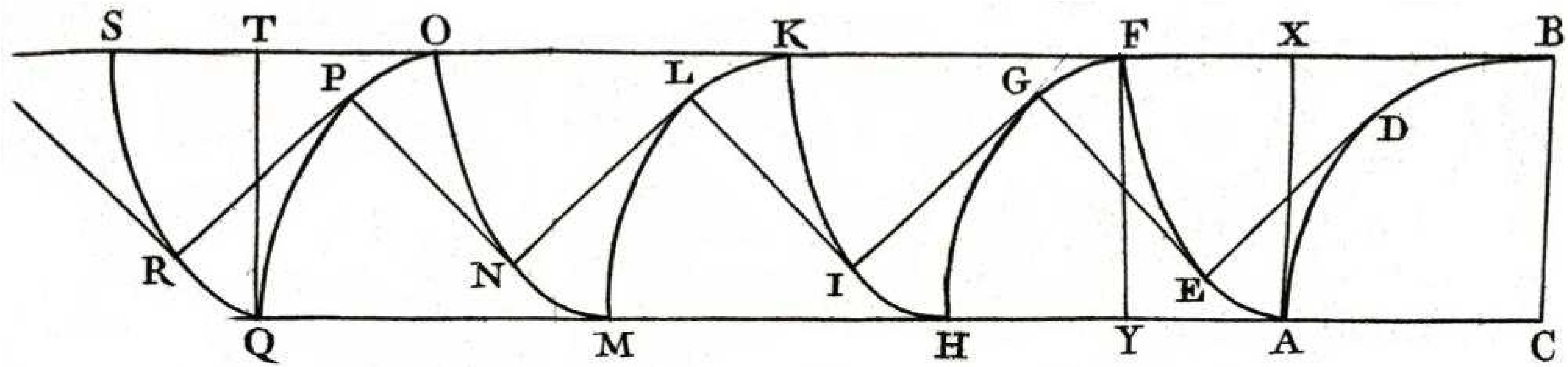


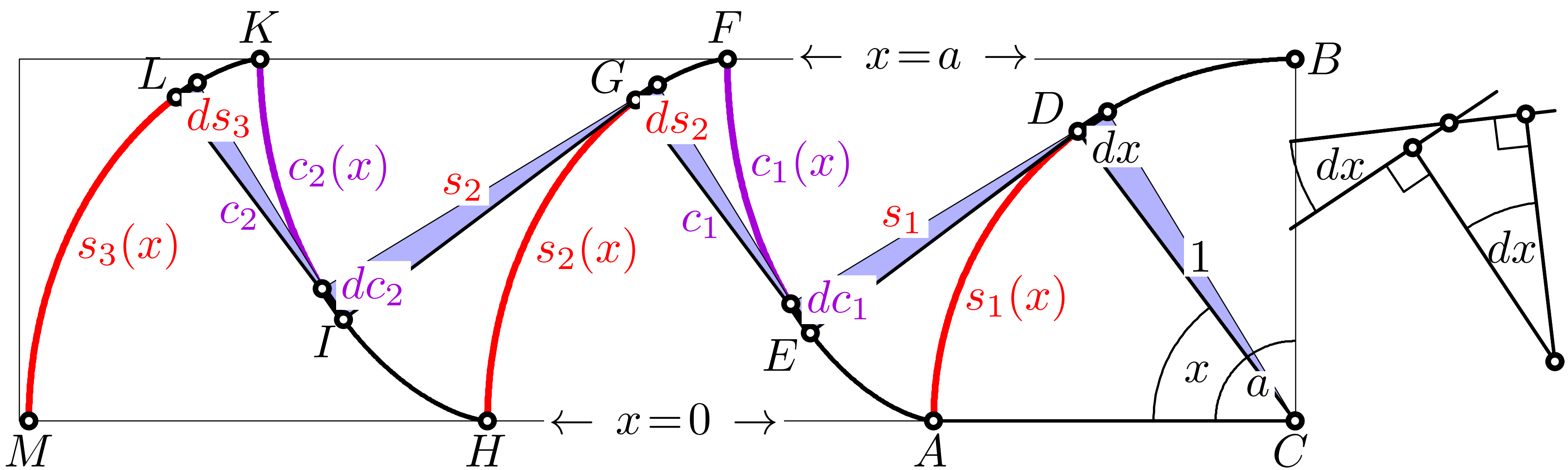
Sit curva quaelibet ADB, cujus tangentes in A et B ad se invicem perpendiculares, atque adeo oppositis coordinatis parallelae; productis itaque axe CA et tangente BF in infinitum; evolvi intelligatur ADB, incipiendo evolutionem in A; quae inde describitur AEF, evolvatur quoque incipiendo a fine F, et inde descripta FGH, porro evolvatur initio facto ab H, et sic fiat in infinitum alterna evolutio, inchoando quamlibet evolutionem a puncto in quo praecedens finitur. Dico post evolutiones in infinitum continuatas, curvas tunc postremo generatas fore cycloides identicas,

Universitätsbibliothek Basel: L Ia 12, 4.1, fol.fol. 245-247

successive involutes **“generatur fore cycloides identices”.**

Formulas for arc lengths (set $a = \frac{\pi}{2}$).

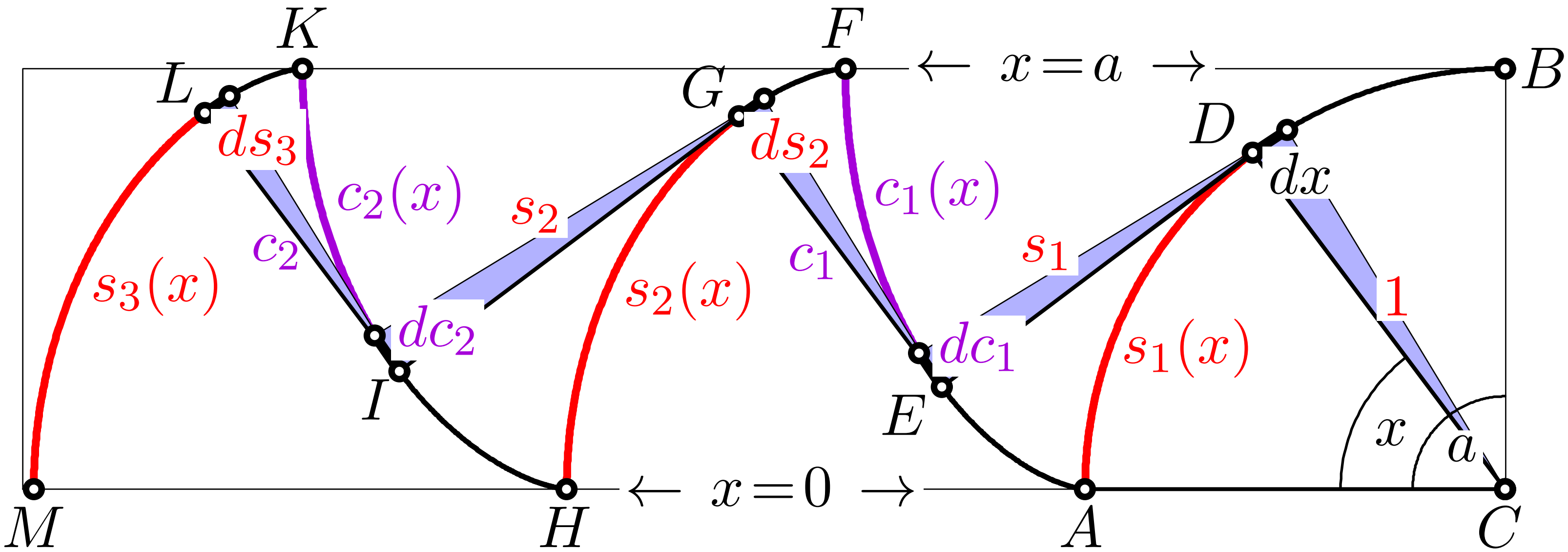




orth. angles ⇒ shaded triangles similar ⇒ $dc_j = s_j \, dx, \; ds_{j+1} = c_j \, dx,$

Bürgi	Bernoulli
$c_{k-\frac{1}{2}} = \sum_{i=k}^{n-1} s_i + \frac{s_n}{2}$	$c_j(x) = \int_x^a s_j(\xi)\, d\xi$
$s'_k = \sum_{i=1}^{k} c_{i-\frac{1}{2}}$	$s_{j+1}(x) = \int_0^x c_j(\xi)\, d\xi$

Computation for the case of a circle of rad 1:

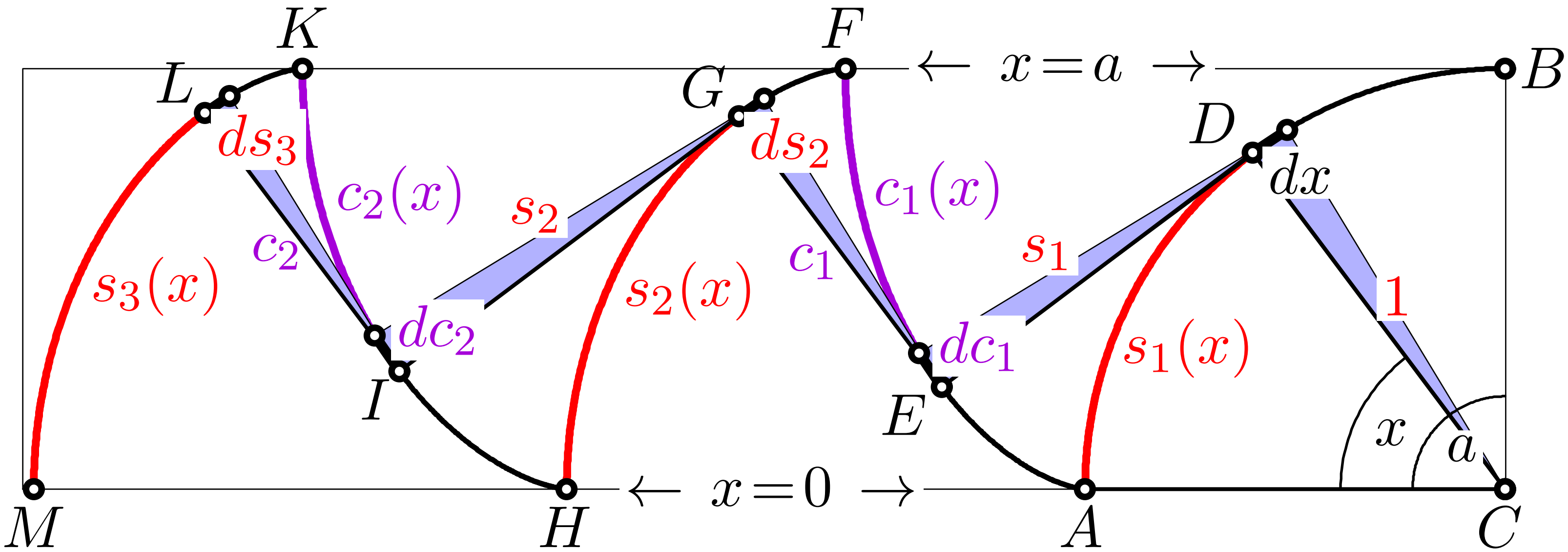


$$s_1(x) = x$$

$$c_1(x) = \int_x^a s_1(\xi)\, d\xi = \frac{a^2}{2!} - \frac{x^2}{2!}, \qquad s_2(x) = \int_0^x c_1(\xi)\, d\xi = \frac{a^2}{2!}x - \frac{x^3}{3!}.$$

From $c_1(0)$ and $s_2(a)$ we obtain $AF = c_1(0) = \boxed{\frac{1a^2}{1\cdot 2}}$ and $FH = s_2(a) = \boxed{\frac{2a^3}{1\cdot 2\cdot 3}}$.

Computation for the case of a circle of rad 1:



$$s_1(x) = x$$

$$c_1(x) = \int_x^a s_1(\xi)\,d\xi = \frac{a^2}{2!} - \frac{x^2}{2!}, \qquad s_2(x) = \int_0^x c_1(\xi)\,d\xi = \frac{a^2}{2!}\,x - \frac{x^3}{3!}.$$

repeat

From $c_1(0)$ and $s_2(a)$ we obtain $AF = c_1(0) = \boxed{\frac{1a^2}{1\cdot 2}}$ and $FH = s_2(a) = \boxed{\frac{2a^3}{1\cdot 2\cdot 3}}$.

$$c_2(0) = \boxed{\frac{5a^4}{1\cdot 2\cdot 3\cdot 4}},\ s_3(a) = \boxed{\frac{16a^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}},\ c_3(0) = \boxed{\frac{61a^6}{1\cdot 2\cdot 3\cdots 6}},\ s_4(a) = \boxed{\frac{272a^7}{1\cdot 2\cdot 3\cdots 7}}, \ldots$$

Johann's result for the arc lengths:

Curva I $= a\,(1)$

.... II $= a^2 \left(\frac{1}{1.2}\right)$

.... III $= a^3 \left(\frac{2}{1.2.3}\right)$

.... IV $= a^4 \left(\frac{5}{1.2.3.4}\right)$

.... V $= a^5 \left(\frac{16}{1.2.3.4.5}\right)$

.... VI $= a^6 \left(\frac{61}{1.2.3\ldots 6}\right)$

.... VII $= a^7 \left(\frac{272}{1.2.3\ldots 7}\right)$

.... VIII $= a^8 \left(\frac{1385}{1.2.3\ldots 8}\right)$

.... IX $= a^9 \left(\frac{7936}{1.2.3\ldots 9}\right)$

.... X $= a^{10} \left(\frac{50521}{1.2.3\ldots 10}\right)$

.... XI $= a^{11} \left(\frac{353792}{1.2.3\ldots 11}\right)$

.... XII $= a^{12} \left(\frac{2702765}{1.2.3\ldots 12}\right)$

.... XIII $= a^{13} \left(\frac{22368256}{1.2.3\ldots 13}\right)$

.... XIV $= a^{14} \left(\frac{199360981}{1.2.3\ldots 14}\right)$

Remarkable sequence of numbers:

1, 1, 1, 2, 5, 16, 61, 272, 1385, 7936, 50521, 353792, 2702765, . . .

are called **"Euler zigzag numbers"** (J.H.Conway, R.Guy)

Johann's *Schediasma cyclometricum*.

Curva I $= a\,(1)$ .... VIII $= a^8\left(\frac{1385}{1.2.3\ldots\ldots8}\right)$

.... II $= a^2\left(\frac{1}{1.2}\right)$ .... IX $= a^9\left(\frac{7936}{1.2.3\ldots\ldots9}\right)$

.... III $= a^3\left(\frac{2}{1.2.3}\right)$ .... X $= a^{10}\left(\frac{50521}{1.2.3\ldots\ldots10}\right)$

.... IV $= a^4\left(\frac{5}{1.2.3.4}\right)$ .... XI $= a^{11}\left(\frac{353792}{1.2.3\ldots\ldots11}\right)$

.... V $= a^5\left(\frac{16}{1.2.3.4.5}\right)$ .... XII $= a^{12}\left(\frac{2702765}{1.2.3\ldots\ldots12}\right)$

.... VI $= a^6\left(\frac{61}{1.2.3\ldots6}\right)$ .... XIII $= a^{13}\left(\frac{22368256}{1.2.3\ldots\ldots13}\right)$

.... VII $= a^7\left(\frac{272}{1.2.3\ldots7}\right)$ .... XIV $= a^{14}\left(\frac{199360981}{1.2.3\ldots\ldots14}\right)$

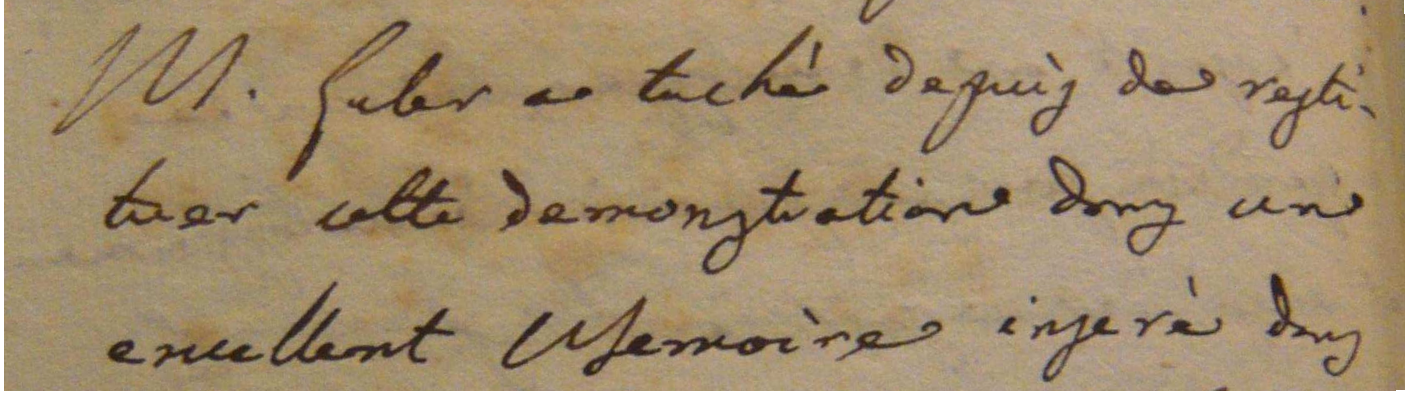
VII=VIII and "dividendo per a^7" $\Rightarrow\ a = \dfrac{8\cdot 272}{1385}$

hence $2a = \pi \simeq \dfrac{16\cdot 272}{1385} = 3.14224$

("nostra analogia tantillo minor est, quam *Archimedea*")

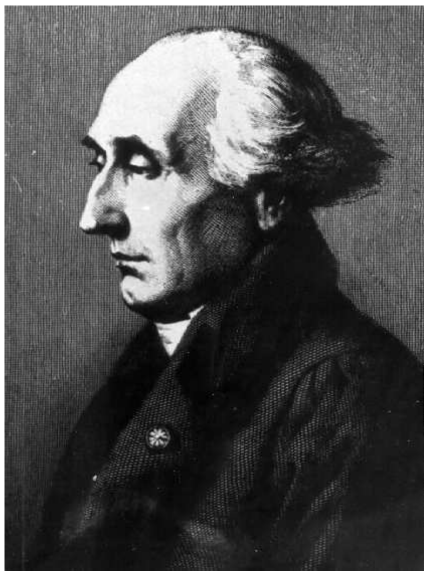
best: XII=XIII and XIII=XIV:

$$\frac{26\cdot 2702765}{22368256} = 3.14159003 < \pi < \frac{28\cdot 22368256}{199360981} = 3.14159353.$$


M. Euler a taché depuis de resti-
tuer cette demonstration dans un
excellent Memoire inseré dans

Lagrange:
M. Euler a taché depuis de restituer cette demonstration dans un excellent Memoire inseré dans ...

4. Leonhard Euler's
"perfectam demonstrationem theorematis BERNOULLIANI"

Buergi 1584 Lagrange 1780 Bernoulli 1742 Euler 1764/78

Euler E300, 1764 (20 pages):

- let $z(x)$ be the "infinitissima curva"
- "nihil enim impedit ..."
  $z = A\sin x + B\sin 3x + C\sin 5x + \ldots$
- "sequentes valores z, z'', z'''', ..."

Lagrange: "mais il me semble que sa methode ne porte pas et ne sauroit porter dans l'esprit toute la lumiere ni toute la conviction qu'on peut desirer sur ce sujet."

- Lagrange: If $z(x)$ were a cycloid, then all derivatives remain the same cycloid, never become $s_2(x)$, $s_1(x)$
  "il faut partir nécessairement de la premiere courbe"
- Later, in 1807, Lagrange will **violently** refuse this point, when **this** man claims it ⇒
- This point will (rightly) be criticized a century later (Abel, Weierstrass,..).

Euler E300, 1764 (20 pages):

- let $z(x)$ be the "infinitissima curva"
- "nihil enim impedit ..."
  $z = A\sin x + B\sin 3x + C\sin 5x + \ldots$
- "sequentes valores z, z'', z'''', ..."

Lagrange: "mais il me semble que sa methode ne porte pas et ne sauroit porter dans l'esprit toute la lumiere ni toute la conviction qu'on peut desirer sur ce sujet."

- Lagrange: If $z(x)$ were a cycloid, then all derivatives remain the same cycloid, never become $s_2(x)$, $s_1(x)$
  "il faut partir nécessairement de la premiere courbe"
- Later, in 1807, Lagrange will **violently** refuse this point, when **this** man claims it ⇒
- This point will (rightly) be criticized a century later (Abel, Weierstrass,..).

Lagrange 1780 First correct proof, 59 pages;

Legendre 1817 Gives short version of Lagrange's proof, saying Euler had given proof and **not** mentioning Lagrange;

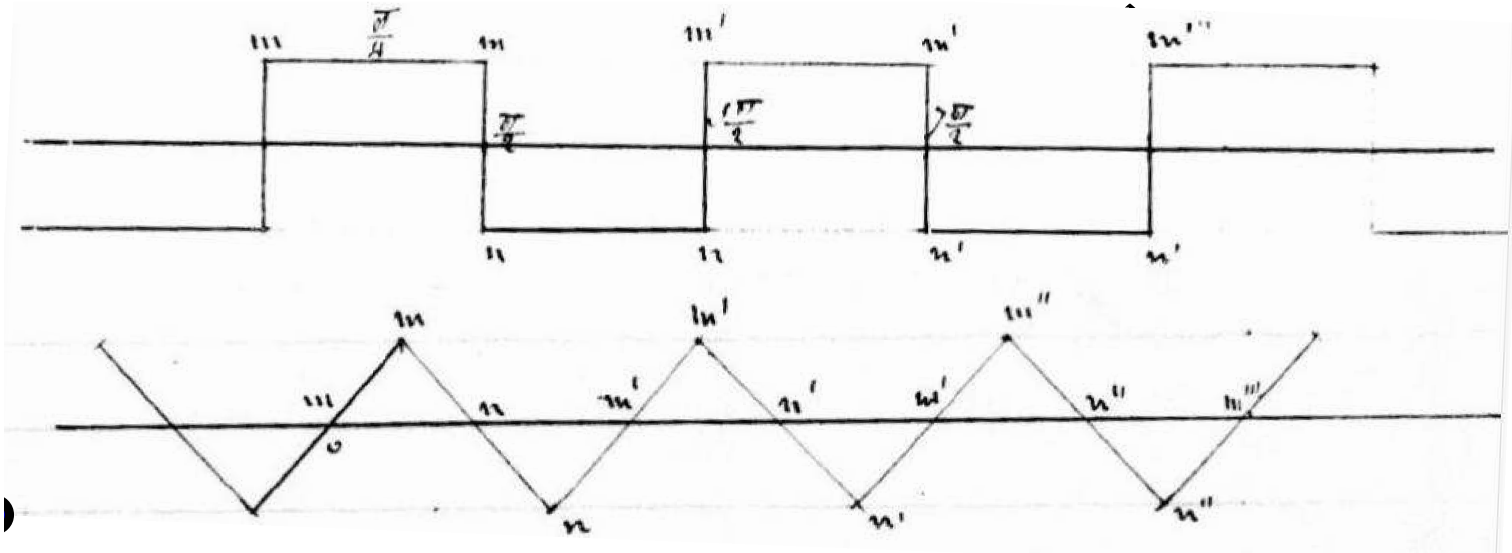
Poisson 1820 (long article "Séries de quantités périodiques" declaring Lagrange inventor of these series, **not** Fourier;

Puiseux 1844 (simplified Poisson's proof). $\Rightarrow$

A century later: Arrive at Euler's proof read backwards ...

- let $s_1(x)$ be the **first** "curva"
- "nihil enim impedit ..."
  $s_1 = A\sin x + B\sin 3x + C\sin 5x + \ldots$
- "sequentes valores s_1, $\int\int s_1$, $\int\int\int\int s_1$, ..."

$c_j(x) = \curvearrowleft \int_x^a s_j(\xi)\, d\xi$	$\curvearrowleft \int_x^{\frac{\pi}{2}} \sin k\xi\, d\xi = \frac{1}{k}\cos kx$
$s_{j+1}(x) = \curvearrowright \int_0^x c_j(\xi)\, d\xi$	$\curvearrowright \int_0^x \cos k\xi\, d\xi = \frac{1}{k}\sin kx$



(Fourier, BNF, Ms. Fr. 22525, fol. 107v)

$$c_0(x) = \frac{4}{\pi}\left(+\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \frac{1}{7}\cos 7x + \dots\right)$$

$$s_1(x) = \frac{4}{\pi}\left(+\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \frac{1}{7^2}\sin 7x + \dots\right)$$

$$c_1(x) = \frac{4}{\pi}\left(+\cos x - \frac{1}{3^3}\cos 3x + \frac{1}{5^3}\cos 5x - \frac{1}{7^3}\cos 7x + \dots\right)$$

$$s_2(x) = \frac{4}{\pi}\Big(+\sin x - \underbrace{\frac{1}{3^4}\sin 3x + \frac{1}{5^4}\sin 5x - \frac{1}{7^4}\sin 7x + \dots}_{\to 0}\Big)$$

we see that

$$\boxed{s_j(x) \to A\sin x}$$

which is characteristic for the cycloid. □

Euler and Bernoulli numbers.

$$s_1\left(\tfrac{\pi}{2}\right) = \tfrac{4}{\pi}\left(+\sin\tfrac{\pi}{2} - \tfrac{1}{3^2}\sin\tfrac{3\pi}{2} + \tfrac{1}{5^2}\sin\tfrac{5\pi}{2} + \ldots\right)$$
$$= \tfrac{4}{\pi}\left(1 + \tfrac{1}{3^2} + \tfrac{1}{5^2} + \ldots\right) = \tfrac{\pi}{2}(\mathbf{1})$$

$$c_1(0) = \tfrac{4}{\pi}\left(+\cos 0 - \tfrac{1}{3^3}\cos 0 + \tfrac{1}{5^3}\cos 0 - \ldots\right)$$
$$= \tfrac{4}{\pi}\left(1 - \tfrac{1}{3^3} + \tfrac{1}{5^3} - + \ldots\right) = \tfrac{\pi^2}{2^2}\left(\tfrac{\mathbf{1}}{1\cdot 2}\right)$$

$$\mathrm{I} = a\,(1)$$
$$\mathrm{II} = a^2\left(\frac{1}{1.2}\right)$$
$$\mathrm{III} = a^3\left(\frac{2}{1.\,2.\,3}\right)$$
$$\mathrm{IV} = a^4\left(\frac{5}{1.\,2.\,3.\,4}\right)$$
$$\mathrm{V} = a^5\left(\frac{16}{1.\,2.\,3.\,4.\,5}\right)$$

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots = \frac{\mathbf{1}\pi}{0!\cdot 2^2}$$
$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \ldots = \frac{\mathbf{1}\pi^3}{2!\cdot 2^4}$$
$$1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{9^5} - \ldots = \frac{\mathbf{5}\pi^5}{4!\cdot 2^6}$$
$$1 - \frac{1}{3^7} + \frac{1}{5^7} - \frac{1}{7^7} + \frac{1}{9^7} - \ldots = \frac{\mathbf{61}\pi^7}{6!\cdot 2^8}$$

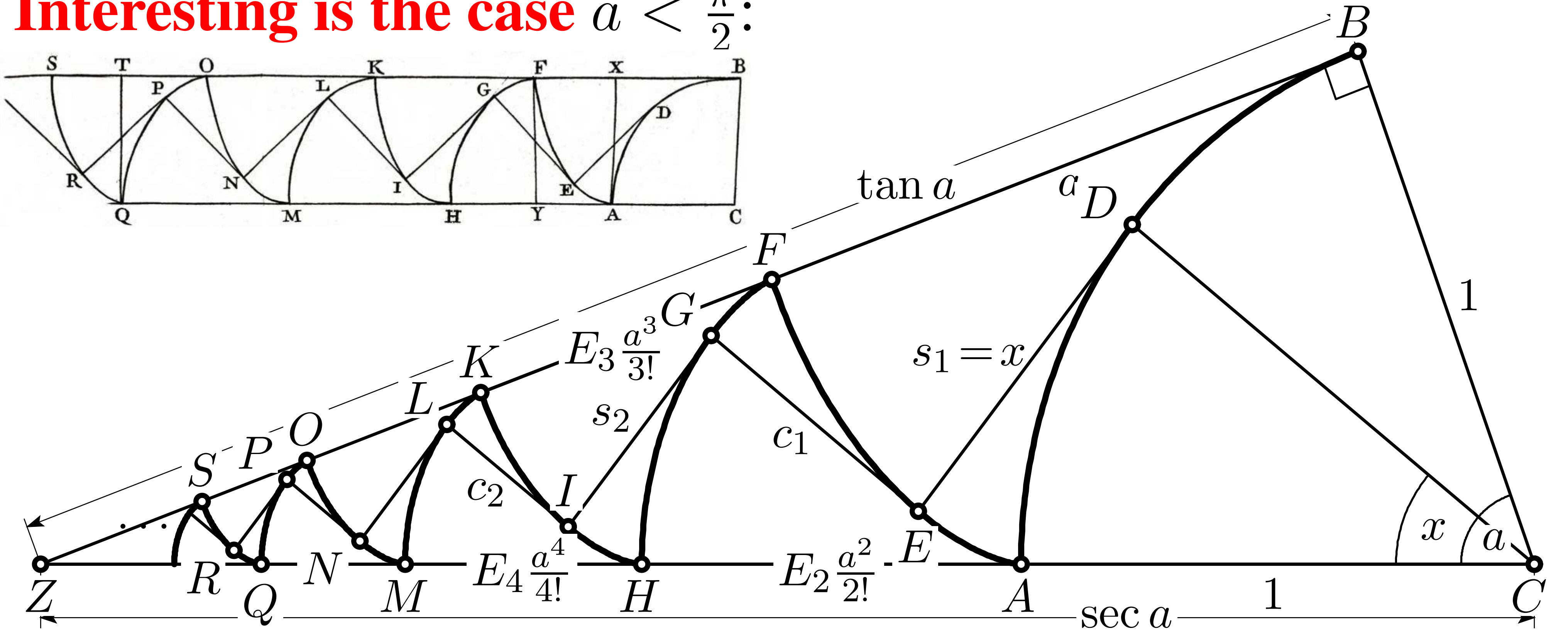
$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \ldots = \frac{\mathbf{1}\pi^2}{1!\cdot 2^3}$$
$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \ldots = \frac{\mathbf{2}\pi^4}{3!\cdot 2^5}$$
$$1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \ldots = \frac{\mathbf{16}\pi^6}{5!\cdot 2^7}$$
$$1 + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \frac{1}{9^8} + \ldots = \frac{\mathbf{272}\pi^8}{7!\cdot 2^9}$$

$$E_0 = 1,\ E_2 = 1,\ E_4 = 5,\ E_6 = 61,\ E_8 = 1385,\ E_{10} = 50521,\ E_{12} = 2702765$$
$$E_1 = 1,\ E_3 = 2,\ E_5 = 16,\ E_7 = 272,\ E_9 = 7936,\ E_{11} = 353792,\ \ldots$$

Appl.: Geometric proof of series for sec and tan:

Johann's calculations $AF = \boxed{\frac{1a^2}{1\cdot 2}} = AH,\ FH = \boxed{\frac{2a^3}{1\cdot 2\cdot 3}} = FK\ldots$ assume nowhere that $a = \frac{\pi}{2}$.

Interesting is the case $a < \frac{\pi}{2}$:

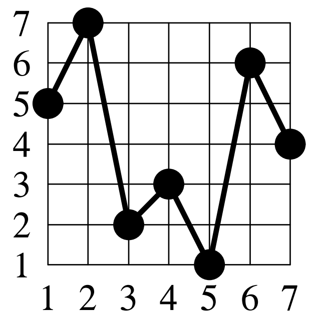


$$\sec a = \frac{1}{\cos a} = \sum_{k=0}^{\infty} \frac{E_{2k}}{(2k)!}\, a^{2k}, \quad \tan a = \sum_{k=0}^{\infty} \frac{E_{2k+1}}{(2k+1)!}\, a^{2k+1}.$$

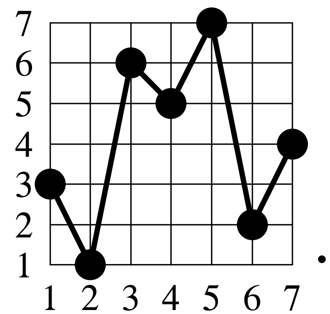
$$\sec a + \tan a = \tan\left(\frac{a}{2} + \frac{\pi}{4}\right) = \sum_{k=0}^{\infty} \frac{E_k}{k!}\, a^k.$$

Connection with Alternating permutations (André 1879)

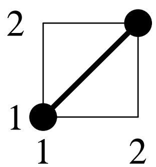
up-down



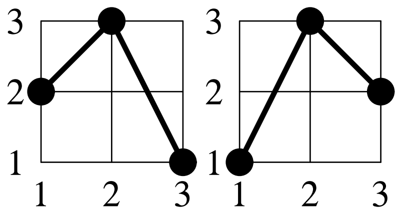
down-up



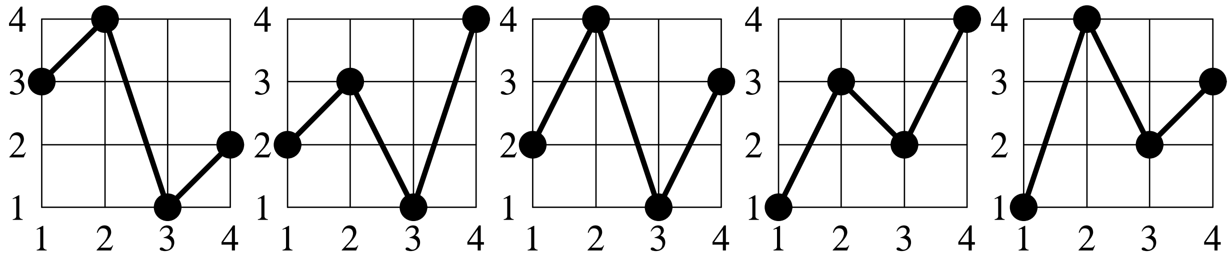
Def. A_n = nr. of (up-down **or** down-up) alt.perm. of n objs.:



$A_2 = 1$



$A_3 = 2$

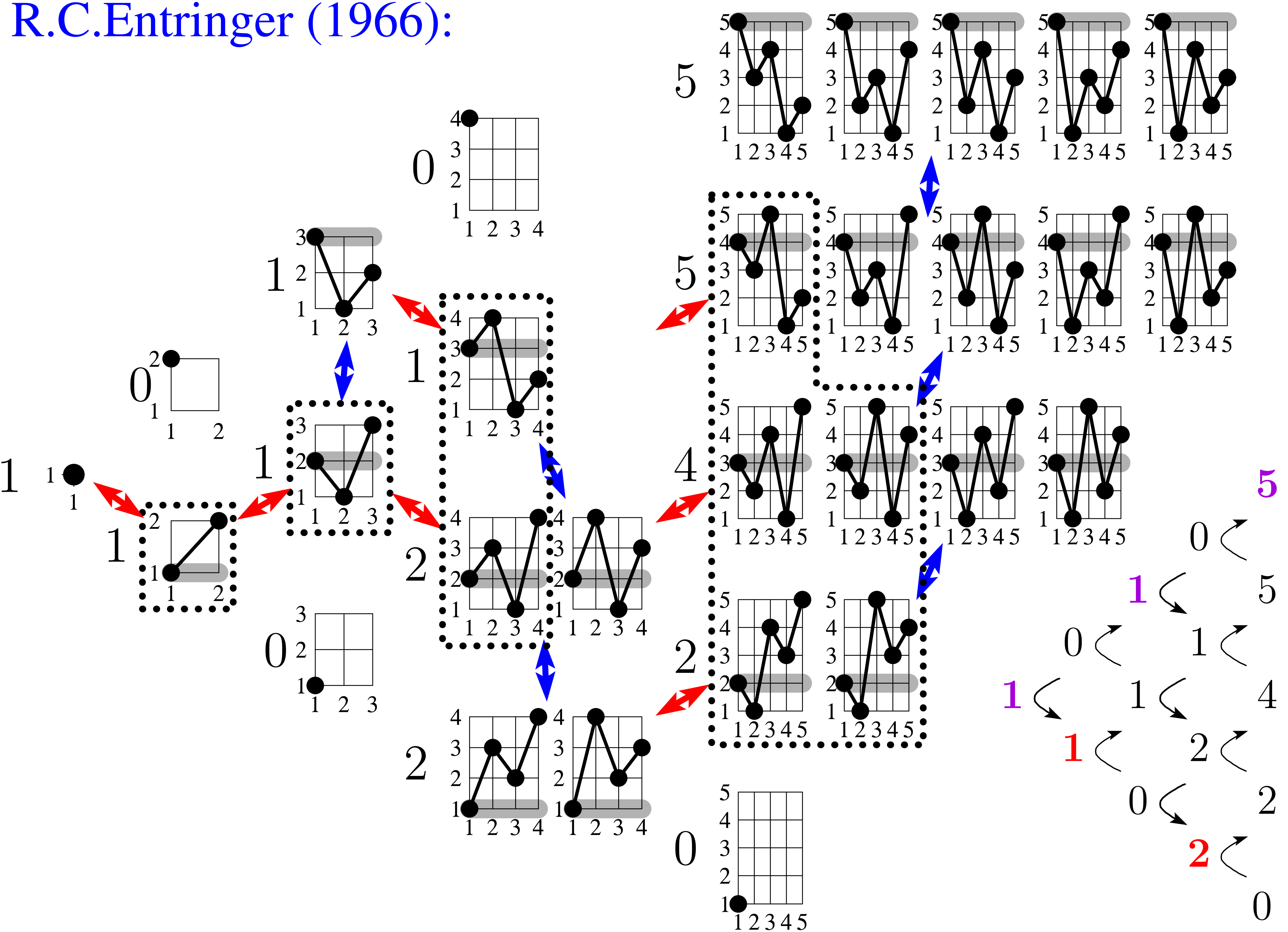


$A_4 = 5$

Theorem. André gives recursive formula for A_n; turn out to be $\equiv E_n$ (Euler zigzag numbers).

$$1, 1, \boxed{1, 2, 5,}\ 16, 61, 272, 1385, 7936, 50521, 353792, 2702765, \ldots$$

R.C.Entringer (1966):



Back to Bürgi – Bernoulli up-down addition !!

V.I.Arnol'd (2000): "J'appelle [ce] triangle de Euler-Bernoulli parce que Pascal ne l'a pas considéré, et parce que Euler et Bernoulli ne l'ont pas considéré non plus"

1	0	**1**	0	**5**	0	**61**	0	**1385**	0	**50521**
	1	1	1	5	5	61	61	1385	1385	50521
		0	2	4	10	56	122	1324	2770	49136
			2	2	14	46	178	1202	4094	46366
				0	16	32	224	1024	5296	42272
					16	16	256	800	6320	36976
						0	272	544	7120	30656
							272	272	7664	23536
								0	7936	15872
									7936	7936
										0

L. Seidel (1877): "in welchem sich wahrscheinlich die einfachste Genesis der B e r n o u l l i ' s c h e n Z a h l e n ausspricht"

(He is the "Gauss-Seidel" Seidel...)

Is now called the SEA-triangle.

Dessert

Euler's E489 ("exponentiales replicatas")

$$r^{r^{r^{r^{\alpha}}}} = ?$$

Example:

$r = 1.5$

$\alpha = 1.5000$

$\beta = 1.5^{\alpha}$
$= 1.8371$

$\gamma = 1.5^{\beta}$
$= 2.1062$

$\delta = 1.5^{\gamma}$
$= 2.3490$

$\epsilon = 1.5^{\delta}$
$= 2.5920$

etc.

$l\alpha = 0{,}1760913$ hincque $\alpha = 1{,}5000$
$llr = 9{,}2457379$

$ll\beta = 9{,}4218292$

$l\beta = 0{,}2641370$ hincque $\beta = 1{,}8371$
$llr = 9{,}2457379$

$ll\gamma = 9{,}5098749$

$l\gamma = 0{,}3235004$ hincque $\gamma = 2{,}1062$
$llr = 9{,}2457379$

$ll\delta = 9{,}5692383$

$l\delta = 0{,}3708841$ hincque $\delta = 2{,}3490$
$llr = 9{,}2457379$
$ll\varepsilon = 9{,}6166220$

$l\varepsilon = 0{,}4136396$ hincque $\varepsilon = 2{,}5920$
$llr = 9{,}2457379$

$ll\zeta = 9{,}6593775$

$l\zeta = 0{,}4564335$ hincque $\zeta = 2{,}8604$
$llr = 9{,}2457379$

$ll\eta = 9{,}7021714$

$l\eta = 0{,}5036993$ hincque $\eta = 3{,}1893$
$llr = 9{,}2457379$

$ll\theta = 9{,}7494372$

$l\theta = 0{,}5616140$ hincque $\theta = 3{,}6443$

"lente increscunt, dubitare ... convergeant"

Euler's E489 ("exponentiales replicatas")

$$r^{r^{r^{r^{\alpha}}}} = ?$$

Example:

$r = 1.5$

$\alpha = 1.5000$

$\beta = 1.5^{\alpha} = 1.8371$

$\gamma = 1.5^{\beta} = 2.1062$

$\delta = 1.5^{\gamma} = 2.3490$

$\epsilon = 1.5^{\delta} = 2.5920$

etc.

$l\alpha = 0,1760913$ hincque $\alpha = 1,5000$
$llr = 9,2457379$

$ll\beta = 9,4218292$

$l\beta = 0,2641370$ hincque $\beta = 1,8371$
$llr = 9,2457379$

$ll\gamma = 9,5098749$

$l\gamma = 0,3235004$ hincque $\gamma = 2,1062$
$llr = 9,2457379$

$ll\delta = 9,5692383$

$l\delta = 0,3708841$ hincque $\delta = 2,3490$
$llr = 9,2457379$
$ll\varepsilon = 9,6166220$

$l\varepsilon = 0,4136396$ hincque $\varepsilon = 2,5920$
$llr = 9,2457379$

$ll\zeta = 9,6593775$

$l\zeta = 0,4564335$ hincque $\zeta = 2,8604$
$llr = 9,2457379$

$ll\eta = 9,7021714$

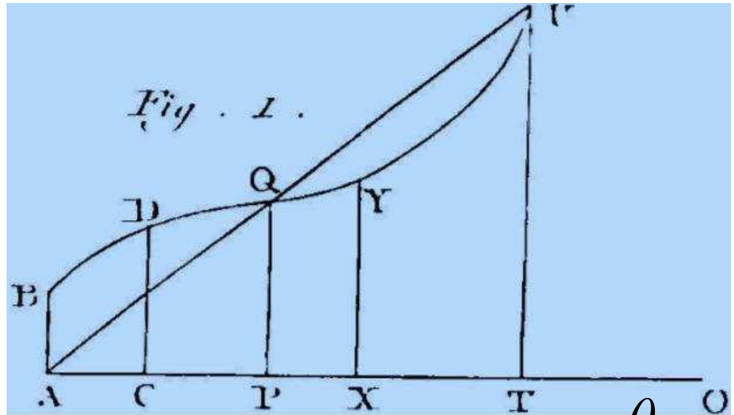
$l\eta = 0,5036993$ hincque $\eta = 3,1893$
$llr = 9,2457379$

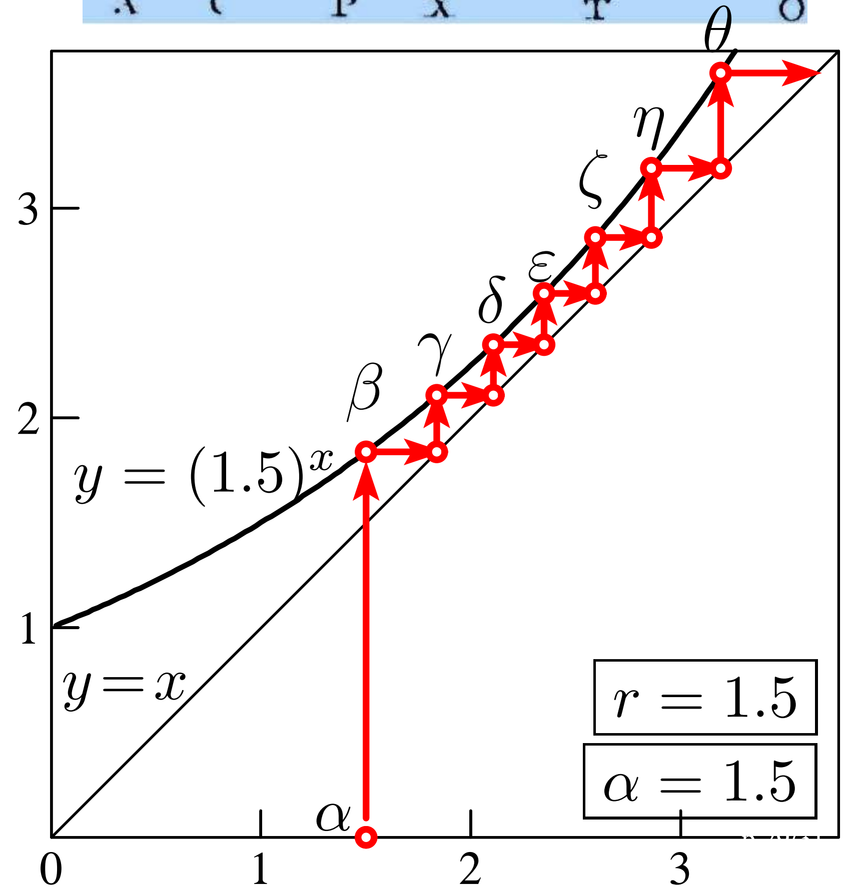
$ll\theta = 9,7494372$

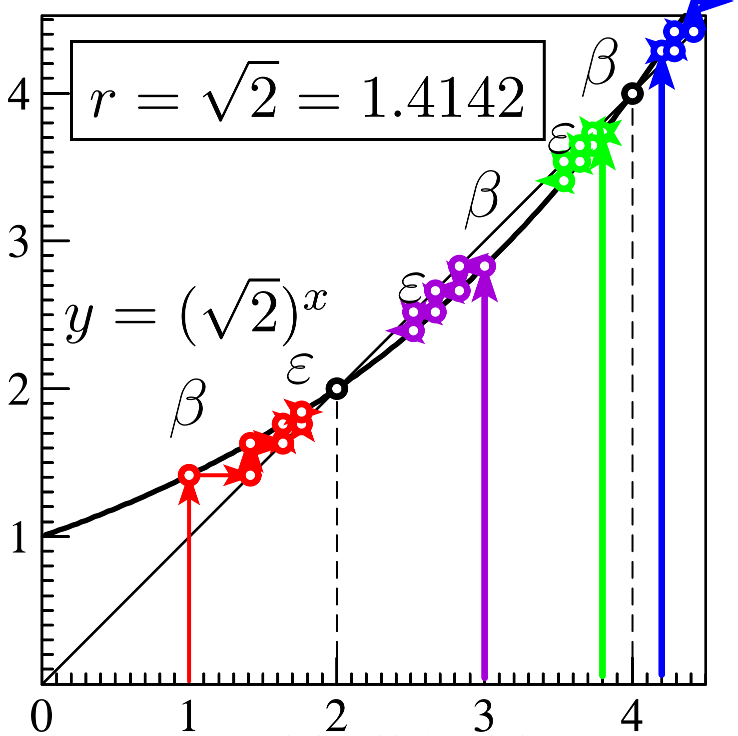
$l\theta = 0,5616140$ hincque $\theta = 3,6443$

"lente increscunt, dubitare ... convergeant"

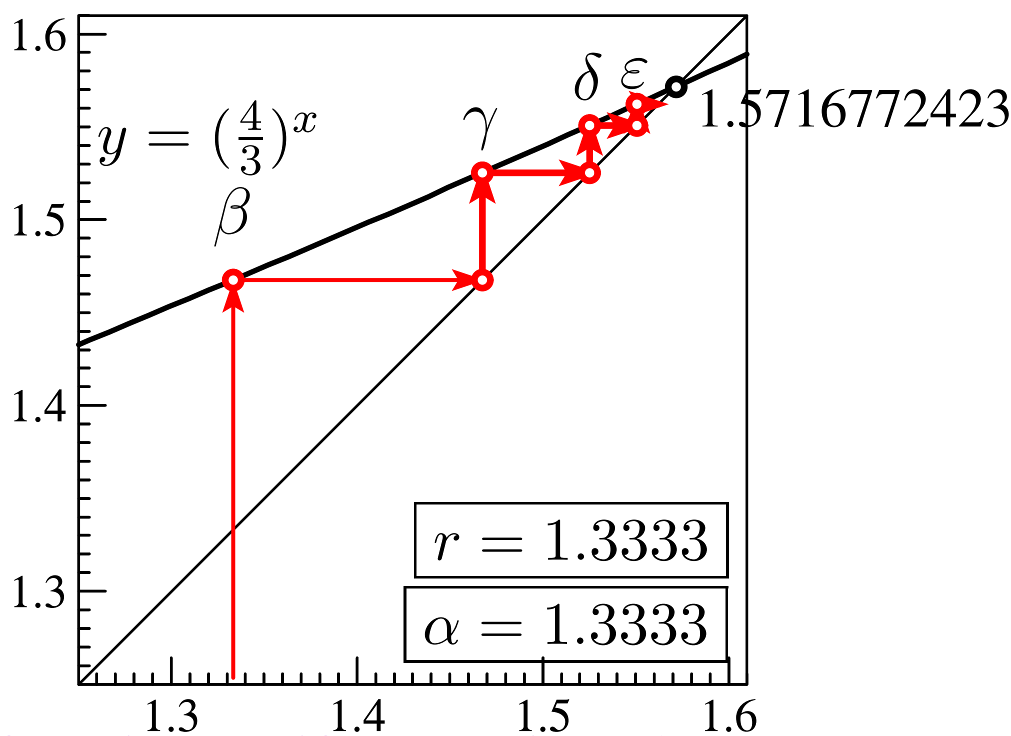
SOLUTIO GEOMETRICA



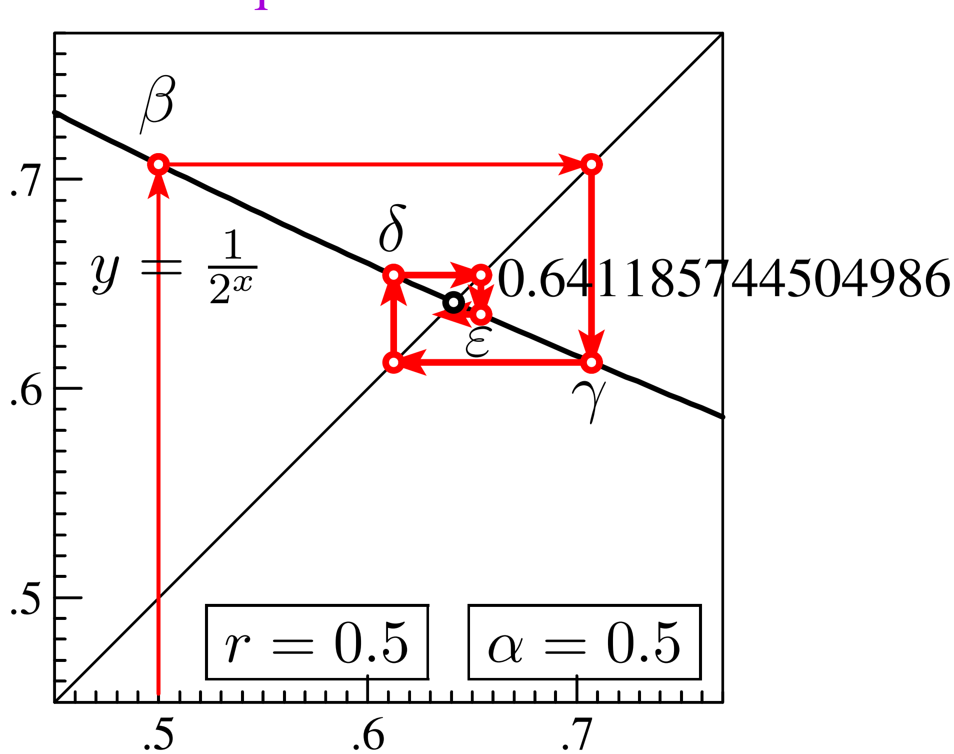




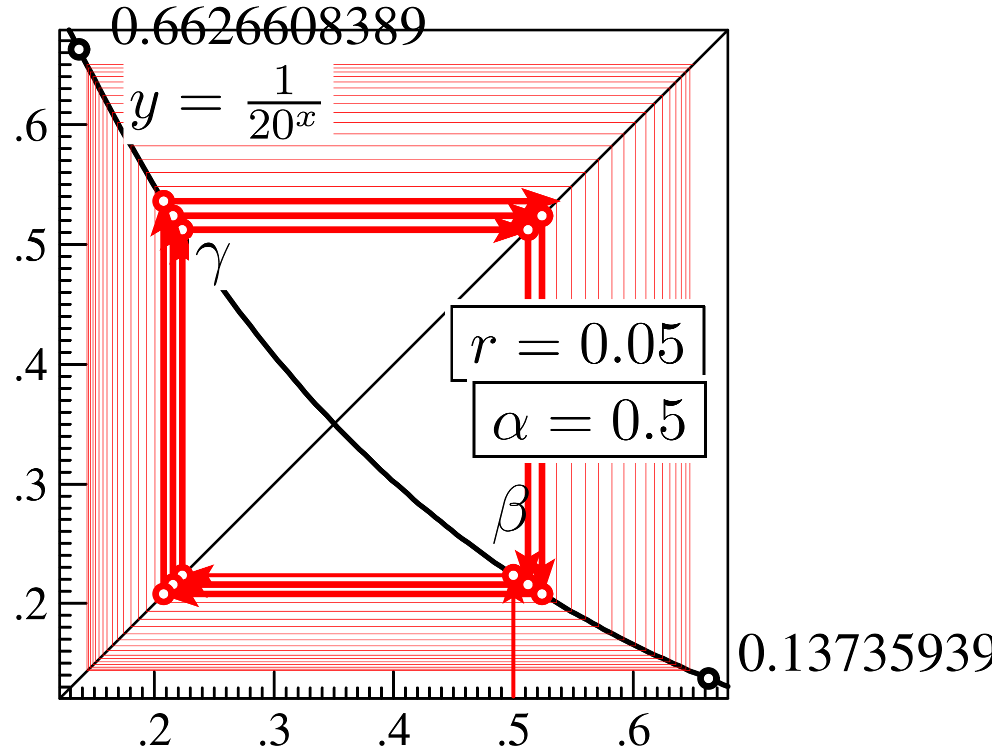
“aequilibrii labilis”



“differentiae manifesto continuo decrescunt”



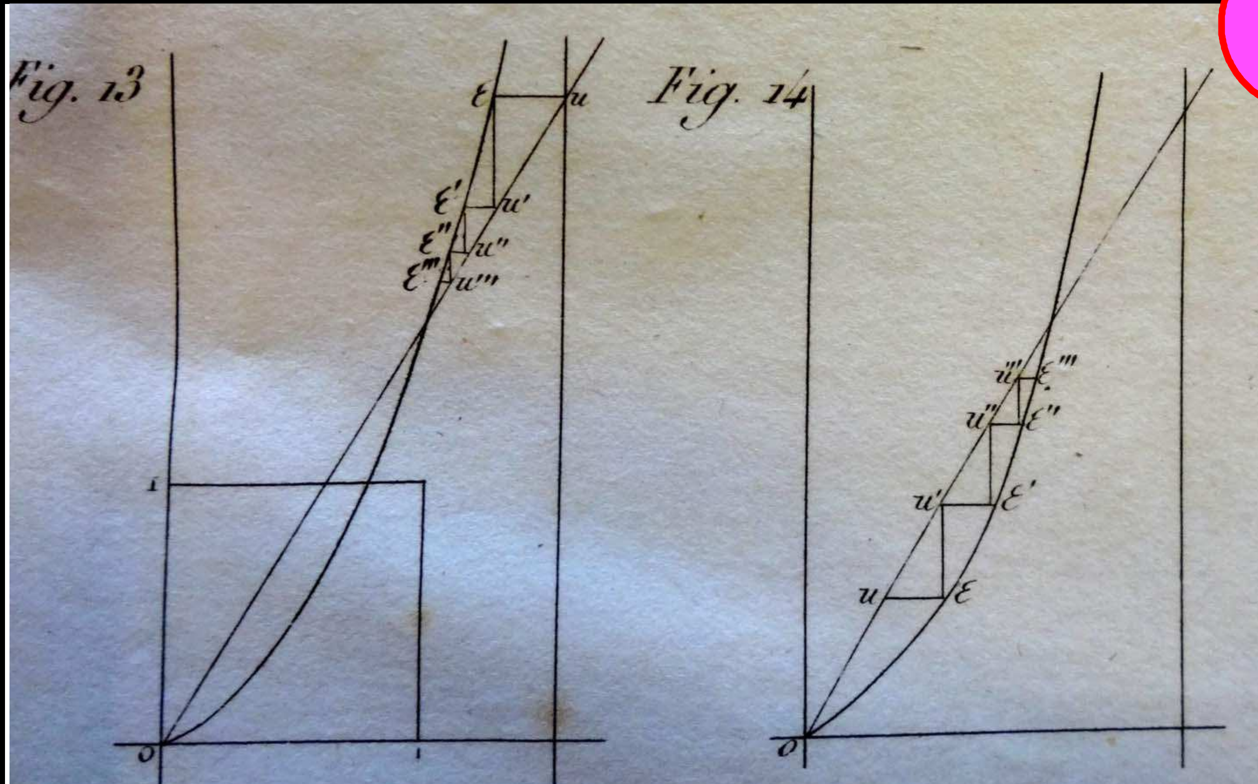
“alternatim superant ab eoque deficiunt”



“alternatim ad duos valores”

who draw the first clear zigzag picture ??

who draw the first clear zigzag picture ??



$$\varepsilon = \ldots \text{arc. tang.}\left(\frac{1}{\lambda}\,\text{arc. tang.}\left(\frac{1}{\lambda}\,\text{arc. tang.}\left(\frac{1}{\lambda}\,\text{arc. tang.}\,\frac{1}{\lambda}\right)\right)\right)$$

(Th. anal. Chaleur, 1822, Chap. V, Art. 286)

Happy 250th Birthday, Joseph Fourier !!

Litteratura:
Ph. Henry, G.W., *Joh.Bernoulli, Posterity*, Elem.Math 2017
Ph. Henry, G.W., *Zigzags*, Elem.Math 2019
Ph. Henry, G.W., *Hommage à Joseph Fourier*, Matapli 119, 2019